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CONTRACTIBLE 4-MANIFOLDS

ALEXANDRA DU

Abstract. Compact, contractible 4-manifolds distinct from $D^4$ were first constructed by Mazur and Poénaru. Sparks defined a collection of compact, contractible 4-manifolds called Jester manifolds. We study Mazur and Jester 4-manifolds. In particular, we show that all Jester manifolds are not homeomorphic to $D^4$, and that a collection of them are pairwise nonhomeomorphic.

1. Introduction

The classification of manifolds is a fundamental problem in geometric topology. It takes on many flavors depending on the dimension and the category—topological (TOP), piecewise linear (PL), or differentiable/smooth (DIFF). We are mainly focused on smooth manifolds of dimension 3 and 4, though we discuss results for other categories and dimensions for perspective. The general classification problem has logical obstructions. Markov [Mar58] showed that for dimension $\geq 4$, the homeomorphism problem for manifolds is in general undecidable (see also Chernavsky and Leksine [CL06] for a modern proof and an overview of further results). Therefore, one typically studies manifolds satisfying some nice properties. For example, smooth, simply-connected 4-manifolds are currently an active area of research.

Even in the extreme case, compact, contractible manifolds turn out to be very rich and interesting. In dimensions $\leq 3$, each compact, contractible manifold is a disk. This is classical in dimensions 1 and 2, and follows from Perelman’s proof of the Poincaré Conjecture in dimension 3. In 1960, Mazur and Poénaru independently constructed compact, contractible 4-manifolds with nonsimply-connected boundaries. Mazur defined a family of such 4-manifolds indexed by $\mathbb{Z}$. We review Mazur’s construction in Section 4. We give four different proofs that the boundary of each Mazur manifold is nonsimply-connected. One proof is completely algebraic. We also prove that the fundamental groups of the boundaries of the Mazur manifolds are pairwise nonisomorphic for $-200 \leq k \leq 200$. It’s probably the case that this holds for all $k \in \mathbb{Z}$, though we have been unable to prove this.
Sparks \cite{Spa18} introduced a new collection of compact, contractible 4-manifolds \(J_k\) where \(k \in \mathbb{Z}\) called Jester manifolds. Sparks’ construction is a variation on Mazur’s construction. Sparks used Jester manifolds to construct countably infinitely many nontrivial compact, contractible 4-manifolds that split as the union of two 4-disks whose intersection is also a 4-disk. Sparks asked whether any \(J_k\) is not homeomorphic to \(D^4\), and whether the \(J_k\)'s represent infinitely many homeomorphism types. We answer Sparks’ first question in the affirmative by proving every \(J_k\) has nonsimply-connected boundary. We also prove that the fundamental groups of the \(\partial J_k\)'s are pairwise nonisomorphic for \(-100 \leq k \leq 100\). Again, this probably holds for all \(k \in \mathbb{Z}\), but that remains open.

This thesis is organized as follows. In Section 2, we review general properties of compact, contractible \(n\)-manifolds and survey their classification over the last century. In Section 3, we review Dehn surgery on 3-manifolds, paying special attention to knots in \(S^3\) and \(S^1 \times S^2\). We also review the Property P and Property R conjectures as well as a pertinent result of Laudenbach \cite{Lau79}. In Section 4, we study Mazur manifolds, and in Section 5, we study Jester manifolds. In Section 6, we close with problems for further study.

We adopt the convention that simply-connected implies path-connected. Manifolds are always Hausdorff and second-countable (meaning there is a countable basis of open sets). In particular, manifolds are metrizable by Urysohn’s metrization theorem. We allow manifolds to have nonempty boundaries. We use singular (co)homology with coefficients in an arbitrary abelian group \(R\); if no coefficient group is indicated, then integer coefficients are implied. Let \(\simeq\) denote homotopy equivalence, \(\approx\) denote \textsc{cat} isomorphism of manifolds, and \(\cong\) denote isomorphism of groups. We use the computational algebra system MAGMA to study finitely presented groups. Note that MAGMA composes permutations left to right.

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2. Compact, Contractible Manifolds

We review some results from algebraic topology relevant to contractible manifolds. For insight, we include geometric proofs for PL and DIFF in addition to general proofs for TOP. A key criterion for recognizing contractibility of a manifold is the following.

Lemma 2.1. Let $X^n$ be an n-manifold. Then, $X^n$ is contractible if and only if $X^n$ is simply-connected and $\tilde{H}_n(X^n) = 0$.

Proof. The forward implication is immediate since $X \simeq pt$. For the converse, Hurewicz’s theorem [Bre93, p. 479] implies the result for CW complexes. The converse follows since each n-manifold is homotopy equivalent to a CW complex (see Hatcher [Hat02, p. 529] for the compact case, and see Milnor [Mil65] for the general case).

Lemma 2.2. Let $X^n$ be a connected n-manifold. Then:

(2.1) $H_k(X; R) = 0$ for $k > n$.
(2.2) if $\partial X^n \neq \emptyset$, then $H_n(X; R) = 0$.
(2.3) if $X^n$ is noncompact, then $H_n(X; R) = 0$.

Proof. Suppose first that $X^n$ is PL. Recall that the simplicial homology of a simplicial complex is isomorphic to singular homology [Bre93, p. 247]. As $X^n$ is an n-complex, the simplicial homology of $X^n$ vanishes in degrees $k > n$. If $X^n$ is compact and $\partial X^n \neq \emptyset$, then $X^n$ strong deformation retracts to an $(n-1)$-complex (use a spanning tree of the dual 1-skeleton as a guide to collapse n-simplices using free faces) and so $H_n(X; R) = 0$. Alternatively, suppose an element of $H_n(X; R)$ is represented by a chain $c = \Sigma c_\sigma \sigma$ where the sum is over the n-simplices of $X^n$ and each $c_\sigma \in R$. As $\partial c = 0$ (for $c$ to be a cycle), we see that $\pm c_\sigma$ is constant over n-simplices, since $X$ is connected. As $\partial X^n \neq \emptyset$, there exists an n-simplex $\sigma$ with a free face, which implies $c_\sigma = 0$. Thus, $c = 0$ and $H_n(X; R) = 0$. If $X^n$ is noncompact, then a similar argument with n-chains implies $H_n(X; R) = 0$ (recall that, by definition, each chain $c = \Sigma c_\sigma \sigma$ has at most finitely many nonzero coefficients).

For the general case where $X^n$ is TOP, $H_k(X; R) = 0$ for $k > n$ by [Bre93, p. 346]. If $X^n$ is noncompact, then $H_n(X; R) = 0$ by [Bre93, p. 346]. If $\partial X^n \neq \emptyset$, then Brown [Bro62] implies that $\partial X^n$ has a collar neighborhood in $X^n$ (see also Connelly [Con71]). Thus, $X^n \simeq \text{Int } X^n$ (by shuffling collars) and so $H_n(X; R) \cong H_n(\text{Int } X; R) = 0$ by the noncompact case.

Lemma 2.3. Let $X^n$ be a connected n-manifold. If $X^n$ is simply-connected, then $H_1(X) = 0$. If $H_1(X) = 0$ or $H_1(X; \mathbb{Z}_2) = 0$, then $X^n$ is orientable. If $X^n$ is orientable, then $\partial X^n$ is orientable.
Proof. The first claim follows since $H_1(X)$ is the abelianization of $\pi_1(X)$.

Next, suppose $X^n$ is nonorientable. There exists the orientable double cover of $X^n$. So $\pi_1(X)$ has an index 2 subgroup, which implies $\pi_1(X)$ surjects to $\mathbb{Z}_2$. All published proofs—of which we are aware—of the existence of the orientable double cover assume $\partial X^n = \emptyset$. We obtain the orientable double cover when $\partial X^n \neq \emptyset$ from these published statements as follows. Let $Y^n$ be $X^n$ union a collar $\partial X^n \times [0, 1)$ glued along $\partial X^n = \partial X^n \times \{0\}$. So, $\partial Y^n = \emptyset$ and $Y^n$ is nonorientable and connected. Therefore, we have $p : \widetilde{Y^n} \to Y^n$ the orientable double cover. As $X^n \subseteq Y^n$, we have the covering map $p| : p^{-1}(X^n) \to X^n$. This is the desired orientable double cover of $X^n$. Alternatively, as we only require an index 2 subgroup of $\pi_1(X^n)$, we may argue as follows. We have $X^n \cong \text{Int } X^n$, so $\pi_1(X) \cong \pi_1(\text{Int } X^n)$. As $\text{Int } X^n$ is nonorientable, its orientable double cover yields an index 2 subgroup of $\pi_1(\text{Int } X^n) \cong \pi_1(X)$.

Abelianizing we get a surjection from $H_1(X)$ to $\mathbb{Z}_2$. So $H_1(X) \neq 0$. The Universal Coefficient theorem \cite{Bre93} p. 283] yields the exact sequence:

$$0 \to H_1(X) \otimes \mathbb{Z}_2 \to H_1(X; \mathbb{Z}_2) \to H_0(X) \ast \mathbb{Z}_2 \to 0$$

We have $H_0(X) \cong \mathbb{Z}$ and $\mathbb{Z} \ast \mathbb{Z}_2 = 0$ \cite{Bre93} p. 278]. By exactness, $H_1(X; \mathbb{Z}_2) \cong H_1(X) \otimes \mathbb{Z}_2$. Therefore:

$$\text{Hom}(H_1(X; \mathbb{Z}_2), \mathbb{Z}_2) \cong \text{Hom}(H_1(X) \otimes \mathbb{Z}_2, \mathbb{Z}_2)$$

$$\cong \text{Hom}(H_1(X), \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2))$$

$$\cong \text{Hom}(H_1(X), \mathbb{Z}_2) \neq 0$$

where the second isomorphism is the tensor-hom adjunction. So $H_1(X; \mathbb{Z}_2) \neq 0$. Summarizing, if $X^n$ is nonorientable, then $H_1(X) \neq 0$ and $H_1(X; \mathbb{Z}_2) \neq 0$. Taking the contrapositive yields the second claim.

If $X^n$ is smooth and $\partial X^n \neq \emptyset$, then one may orient $\partial X^n$ using the outward normal first convention \cite{GP10} p. 97]. If $X^n$ is $\text{TOP}$, then see \cite{Bre93} p. 356] and \cite{Hat02} p. 260].

Lemma 2.4. Let $X^n$ be a compact, connected $n$-manifold. Then, $\partial X^n \neq \emptyset$ if and only if $H_n(X; \mathbb{Z}_2) = 0$.

Proof. The forward implication holds by Lemma 2.2. For the converse, we prove the contrapositive. First, consider the case where $X$ is $\text{PL}$ and $\partial X^n = \emptyset$. Then $c = \Sigma \sigma$ (the sum of all $n$-simplices) has $\partial c = 0$, so $0 \neq [c] \in H_n(X; \mathbb{Z}_2)$. In general, \cite{Bre93} p. 346] implies that $H_n(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$. □
The next proposition gives a fundamental property of compact, contractible manifolds.

**Proposition 2.5.** If $X^n$ is a compact, contractible $n$-manifold, then $\partial X^n$ is an integral homology $(n - 1)$-sphere.

**Proof.** By Lemma 2.1, $X^n$ is simply-connected and $\tilde{H}_*(X^n) = 0$. By Lemma 2.3, $X^n$ is orientable. The Universal Coefficient theorem [Bre93, p. 282] implies that $\tilde{H}^*(X^n) = 0$. So, we have the following diagram [Bre93, p. 357]; its rows are the long exact sequences of the pair $(X^n, \partial X^n)$, the diagram commutes up to sign, and the vertical isomorphisms are Lefschetz and Poincaré Duality.

$$
\begin{array}{cccccc}
\rightarrow & H^{n-k-1}(X) & \rightarrow & H^{n-k-1}(\partial X) & \rightarrow & H^{n-k}(X, \partial X) & \rightarrow & H^{n-k}(X) & \rightarrow \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \\
\rightarrow & H_{k+1}(X, \partial X) & \rightarrow & H_k(\partial X) & \rightarrow & H_k(X) & \rightarrow & H_k(X, \partial X) & \\
\end{array}
$$

If $n = 1$, then we get $H_k(\partial X)$ has rank 2 for $k = 0$ and vanishes otherwise. If $n > 1$, then we get $H_k(\partial X)$ is isomorphic to $\mathbb{Z}$ for $k = 0$ and $k = n - 1$, and vanishes otherwise.

The next result is a basic criterion for recognizing contractibility of a compact 3-manifold.

**Proposition 2.6.** Let $X^3$ be a compact, simply-connected 3-manifold with connected, nonempty boundary $\partial X^3$. Then $X^3$ is contractible.

**Proof.** By Lemmas 2.2 and 2.3, $H_0(X) \cong \mathbb{Z}$, $H_1(X) = 0$, and $H_k(X) = 0$ for $k \geq 3$. As in Proposition 2.5, we have the following diagram [Bre93, p. 357].

$$
\begin{array}{cccccc}
\rightarrow & H^0(X) & \rightarrow & H^0(\partial X) & \rightarrow & H^1(X, \partial X) & \rightarrow & H^1(X) & \rightarrow \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \\
\rightarrow & H_3(X, \partial X) & \rightarrow & H_2(\partial X) & \rightarrow & H_2(X) & \rightarrow & H_2(X, \partial X) & \\
\end{array}
$$

As is well-known [Bre93, p. 356], $\partial_*$ is an isomorphism between copies of $\mathbb{Z}$. By exactness, $i_* = 0$. The Universal Coefficient theorem implies $H^1(X) = 0$. By exactness, $H_2(X) = 0$. Hence, $X^3$ is contractible by Lemma 2.1.

**Remark 2.7.** In Proposition 2.6, the simply-connected hypothesis cannot be replaced with trivial first integral homology. Indeed, there exist nontrivial integral homology 3-disks. Simply remove a nice, open 3-disk from any nontrivial integral homology 3-sphere.
Remark 2.8. Proposition 2.6 is false for 4-manifolds. For example, let $X^4$ be a 0-handle union a 2-handle. Then $X^4 \simeq S^2$, so $X^4$ is a compact, simply-connected 4-manifold with connected, nonempty boundary. However, $X^4$ is not contractible since $H_2(X^4) \cong \mathbb{Z}$. For an explicit example, let $X^4$ be obtained by attaching the 2-handle along a $-1$-framed left-handed trefoil knot. Then $\partial X^4$ is the Poincaré homology 3-sphere, which does not bound any contractible 4-manifold, as a consequence of Rohlin’s invariant [GS99, p. 193]. In fact, every connected, closed, orientable 3-manifold arises as the boundary of a compact, simply-connected 4-manifold obtained by adding finitely many 2-handles to a 0-handle [GS99, p. 159].

Remark 2.9. In Proposition 2.6 we may also conclude that $\partial X^3 \cong S^2$. As $X^3$ is contractible, Proposition 2.5 implies that $\partial X^3$ is an integral homology 2-sphere. So, $\partial X^3 \cong S^2$ by the classification of closed surfaces. Alternatively, we may use the next lemma which is proved by a classic Euler characteristic argument and the classification of closed surfaces.

Lemma 2.10. Let $X^3$ be a compact, connected 3-manifold with $H_1(X) = 0$ or $H_1(X; \mathbb{Z}_2) = 0$. Then $\partial X^3$ is a disjoint union of finitely many (possibly zero) 2-spheres.

Proof. By Lemma 2.3 $\partial X^3$ is a finite disjoint union of closed, orientable surfaces. Suppose, by way of contradiction, that $X^3$ has at least one boundary component that is not a 2-sphere. Let $X'$ be $X^3$ with each 2-sphere boundary capped with a $D^3$. So, $X'$ is a compact 3-manifold and $\partial X'$ is a nonempty finite disjoint union of closed, orientable surfaces each of genus $\geq 1$. In particular, $\chi(\partial X') \leq 0$. Let $DX'$ be the double of $X'$, which is a closed, orientable 3-manifold. Then:

$$0 = \chi(DX') = \chi(X') + \chi(X') - \chi(\partial X') = 2\chi(X') - \chi(\partial X')$$

where the first equality holds by Poincaré Duality, and the second equality holds by the inclusion-exclusion principle. Thus $\chi(\partial X') = 2\chi(X')$. We also have:

$$\chi(X') = \text{rank} H_0(X') - \text{rank} H_1(X') + \text{rank} H_2(X') - \text{rank} H_3(X')$$

$$= 1 - 0 + \text{rank} H_2(X') - 0 \geq 1$$

by Lemma 2.2. Therefore, $0 \geq \chi(\partial X') = 2\chi(X') \geq 2$, a contradiction. The same argument works for $\mathbb{Z}_2$ coefficients with $\text{rank} H_k(X')$ replaced by $\text{dim} H_k(X'; \mathbb{Z}_2)$. Simply note that the Mayer-Vietoris sequence holds for $\mathbb{Z}_2$ coefficients (so $H_1(X'; \mathbb{Z}_2) = 0$) and recall that:

$$\chi(X') = \sum_k (-1)^k \text{rank} H_k(X') = \sum_k (-1)^k \text{dim} H_k(X'; \mathbb{Z}_2)$$

$\square$
Having discussed general algebraic topological properties of compact, contractible manifolds, we close this section with a brief survey of results from the last century on their CAT classification.

In dimensions \( n \leq 3 \), every TOP manifold admits an essentially unique PL and DIFF structure. For PL, this was shown for \( n = 2 \) in 1924 by Radó [Rad24] and for \( n = 3 \) in 1952 by Moise [Moi77, p. v]. For DIFF, this was shown in 1960 by Munkres [Mun60, §6] (see also Hatcher [Hat13] and Thurston [Thu97, §3.10]). There is a unique compact, contractible manifold in dimensions 1 and 2 by the classification of 1- and 2-manifolds (for \( n = 1 \), see Gale [Gal87], Milnor [Mil97, App.], and Guillemin and Pollack [GP10, App. 2]; for \( n = 2 \), see Stillwell [Sti80, p. 69], Moise [Moi77, p. 155], and Hirsch [Hir76, p. 205]).

The first examples of nontrivial compact, contractible manifolds were found independently in 1960 by Mazur and Poénaru [Maz61, Poe60]. These are 4-manifolds whose boundaries are nonsimply-connected integral homology 3-spheres in accordance with Proposition 2.5. We study Mazur’s construction in Section 4 ahead. Briefly, Mazur started with a 0-handle union a 1-handle, and added a 2-handle that algebraically cancelled the 1-handle.

In 1969, Kervaire [Ker69] showed that: (i) there exist nontrivial homology \( n \)-spheres for each \( n \geq 4 \), (ii) every homology 4-sphere bounds a smooth contractible manifold, (iii) every PL homology \( n \)-sphere with \( n \neq 3 \) bounds a PL contractible manifold, and (iv) every PL homology sphere admits a smooth structure.

Smale proved the h-cobordism theorem around 1960. It follows that smooth, compact, contractible \( n \)-manifolds with simply-connected boundaries are diffeomorphic to \( D^n \) for \( n \geq 6 \) (see Milnor [Mil65, p. 108]). Furthermore, if \( X^5 \) is a compact, simply-connected, smooth 5-manifold with \( H_\ast(X^5) \cong H_\ast(pt) \) and \( \partial X^5 \) is diffeomorphic to \( S^4 \), then \( X^5 \) is diffeomorphic to \( D^5 \) (see Milnor [Mil65, p. 110]). Similar results were proven for PL by Stallings and Zeeman and for TOP by Newman [Sta65, Zee61, New66].

Work of Freedman and Quinn up to 1990 showed that each integral homology 3-sphere bounds a TOP contractible 4-manifold that is unique up to homeomorphism relative to the boundary. In particular, \( D^4 \) is the unique compact, contractible TOP 4-manifold with boundary \( S^3 \). For a modern reference, see [BKKPR21, pp. 303–304].

Perelman’s work up to 2003 proved the 3-dimensional Poincaré Conjecture [MT07, Ch. 18]. As a corollary, we have: if \( X^3 \) is a compact, simply-connected 3-manifold
with connected, nonempty boundary $\partial X^3$, then $X^3 \approx D^3$. To see this, note that $\partial X^3 \approx S^2$ by Remark 2.9. So, $\Sigma^3 = X^3 \cup_{\partial} D^3$ is a closed, simply-connected 3-manifold. By Perelman, $\Sigma^3 \approx S^3$. By ambient uniqueness of 3-disk embeddings in the interior of a connected 3-manifold (here $S^3$), we get $X^3 \approx D^3$.

This vast medley of theorems and techniques shows that if $X^n$ is a compact, contractible CAT $n$-manifold and $\partial X^n \approx S^{n-1}$, then $X^n \approx D^n$—with one possible exception: dimension $n = 4$ where PL = DIFF. An exotic 4-ball is a DIFF 4-manifold homeomorphic but not diffeomorphic to $D^4$. The question remains: does there exist an exotic 4-ball? This is equivalent to the (still open) DIFF 4-dimensional Poincaré Conjecture: there does not exist an exotic 4-sphere. The equivalence of these two questions follows from Cerf’s theorem [Cer68] (which proves that every diffeomorphism of $S^3$ is isotopic to the identity or a reflection) and ambient uniqueness of 4-disk embeddings in the interior of a connected DIFF 4-manifold. The existence of an exotic 4-ball splits into two questions depending on whether or not the interior of the hypothetical exotic 4-ball is standard (meaning diffeomorphic to $\mathbb{R}^4$):

(2.1) The DIFF 4-dimensional Schoenflies conjecture: every DIFF embedded $S^3$ in $\mathbb{R}^4$ bounds a DIFF copy of $D^4$.

(2.2) Does there exist an exotic $\mathbb{R}^4$ that is DIFF collared at infinity by $S^3$?

In every dimension $n \neq 4$, each DIFF $n$-manifold homeomorphic to $\mathbb{R}^n$ is necessarily diffeomorphic to $\mathbb{R}^n$. In stark contrast, there exist uncountably many nondiffeomorphic DIFF 4-manifolds homeomorphic to $\mathbb{R}^4$ [GS99, p. 370].

3. Dehn Surgery

In this section, we review Dehn surgery on PL and DIFF 3-manifolds. Let $M^3$ be a 3-manifold whose boundary contains a torus $T$. Let $J$ be an essential simple closed curve on $T$. The manifold $N^3$ is obtained by Dehn filling $M^3$ along $J$ provided $N^3$ is $M^3$ union a 2-handle attached along $J$ and union a 3-handle attached to the resulting 2-sphere. Equivalently, $N^3$ is obtained by gluing $S^1 \times D^2$ to $M^3$ so that $\{pt\} \times \partial D^2$ glues to $J$.

Let $K$ be a knot in the interior of an orientable 3-manifold $M^3$, and let $\nu K \approx S^1 \times D^2$ be a tubular neighborhood of $K$ in $\text{Int } M^3$. The knot complement is $M^3 - K$. The knot exterior is $X = M^3 - \text{Int } \nu K$ with a distinguished boundary torus $T = \partial \nu K$. Let $J$ be an essential simple closed curve on $T$. If $N^3$ is obtained by Dehn filling $X$ along $J$, then we say that $N^3$ is obtained by a Dehn surgery on $K$ in $M^3$. The Dehn surgery is trivial provided $J = \{p\} \times \partial D^2$ under the identification $\nu K \approx S^1 \times D^2$; in this case $N^3 \approx M^3$. 
Independently, Wallace and Lickorish proved that every closed, connected, orientable 3-manifold may be obtained by Dehn surgery on some link in $S^3$ [Wa60, Lic62]. A general—but difficult—question to keep in mind is: given a 3-manifold $M^3$, which manifolds may be generated by performing Dehn surgery on a single knot in $M^3$? Several other questions arise, including uniqueness.

Let $\Sigma^3$ be an oriented integral homology 3-sphere, and let $K$ be an oriented knot in $\Sigma^3$. Let $\nu K \approx S^1 \times D^2$ be a tubular neighborhood of $K$ in $\Sigma^3$. Let $X = \Sigma^3 - \text{Int} \nu K$ be the knot exterior. Let $\lambda$ and $\mu$ be a standard longitude and meridian of $K$ on $\partial \nu K$. Recall that $\lambda$ and $\mu$ are oriented simple closed curves essential on $\partial \nu K$, $\lambda$ bounds a Seifert surface in $X$ (in particular, $0 = [\lambda] \in H_1(X)$), $\mu$ generates $H_1(X) \cong \mathbb{Z}$, and $H_1(\partial \nu K) \cong \mathbb{Z} \times \mathbb{Z}$ where $\lambda$ and $\mu$ form the canonical basis. See Lickorish [Lic97, pp. 11–13] for the standard homology calculation using the Mayer-Vietoris sequence. Choose an essential simple closed curve $J$ on $\partial X = \partial \nu K$. We have $J = p\mu + q\lambda$ where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ (the cases $p = 0$ and $q = \pm 1$, as well as $q = 0$ and $p = \pm 1$, are allowed). Then, $p/q$ Dehn Surgery on $K$ is defined to be the closed, oriented 3-manifold $\Sigma'$ obtained by gluing $S^1 \times D^2$ to $X$ such that $\{pt\} \times \partial D^2$ is glued to $J$. Observe that $H_1(\Sigma') \cong \mathbb{Z}/p\mathbb{Z}$. Here are some canonical examples:

(3.1) $\infty$-surgery on $K$ gives $\Sigma' \approx \Sigma^3$. Here, $p = \pm 1$ and $q = 0$. This is the trivial surgery.

(3.2) 0-surgery on $K$ gives $H_1(\Sigma') \cong \mathbb{Z}$. Here, $p = 0$ and $q = \pm 1$. By Poincaré Duality and the Universal Coefficient theorem, $\Sigma'$ is an integral homology $S^1 \times S^2$. If $K$ is the unknot in $S^3$, then $\Sigma' \approx S^1 \times S^2$.

(3.3) $\pm 1/q$-surgery gives an integral homology 3-sphere $\Sigma'$. Here $p = \pm 1$ and $q \in \mathbb{Z}$.

(3.4) If $\Sigma = S^3$, $K$ is the unknot, and $\Sigma'$ is obtained by performing $p/q$ surgery on $K$, then $\pi_1(\Sigma') \cong \mathbb{Z}/p\mathbb{Z}$.

**Remark 3.1.** It is important to note two subtleties here. First, in case $\Sigma^3$ is an oriented integral homology 3-sphere, the ratio $p/q$ alone suffices to specify the Dehn surgery. Second, for a general knot $K$ in the interior of a general 3-manifold $M^3$, there are no unique choices of (oriented) meridian or longitude. We may take a meridian of $K$ to be the boundary of a 2-disk whose interior meets $K$ transversely at a single point, and then an orientation of $K$ and $M^3$ does uniquely specify an oriented meridian $\mu$. Even in the case where $K$ and $M^3$ are oriented, there is no unique choice of longitude. Indeed, consider the example where $K \subset S^1 \times S^2$ is obtained by tying a small knot in $S^1 \times \{pt\}$ (in particular, $0 \neq [K] \in H_1(S^1 \times S^2)$).

**Claim 3.2.** For two knots $K$ and $K'$ in a closed, orientable manifold $M^3$, the following are equivalent:

(3.1) (Exteriors are homeomorphic) There exists a homeomorphism $g : X \to X'$ where $X$ and $X'$ denote the knot exteriors of $K$ and $K'$. 
(3.2) (Complements are homeomorphic) There exists a homeomorphism $f : M^3 - K \to M^3 - K'$.

Furthermore, these two equivalent statements are implied by the following.

(3.3) (Knots are equivalent) There exists a homeomorphism $h : (M^3, K) \to (M^3, K')$.

Proof. For $[3.1] \Rightarrow [3.2]$ and $X' \approx X''$ implies $X - \partial X \approx X' - \partial X'$. By reparametrizing collars, $X - \partial X \approx M^3 - K$ and $X' - \partial X' \approx M^3 - K'$. Hence, $M^3 - K \approx M^3 - K'$. The implication $[3.2] \Rightarrow [3.1]$ follows by concentric tori (see Edwards [Edw64 Thm. 1] or Brown [Bro66 Thm. 7.2]). Finally, $[3.3] \Rightarrow [3.2]$ by restricting $h$.

Claim 3.3. For a knot $K$ in $S^3$ with exterior $X$, the following are equivalent.

(3.1) $K$ is the unknot.
(3.2) $X$ is homeomorphic to $S^1 \times D^2$.
(3.3) $\pi_1(X) \cong \mathbb{Z}$.

Proof. The implications $[3.1] \Rightarrow [3.2] \Rightarrow [3.3]$ are straightforward. The implication $[3.3] \Rightarrow [3.1]$ is a well-known corollary of Dehn’s Lemma and the Loop Theorem as proved by Papakyriakopoulos (see [Rol76 p. 103]). The implication $[3.2] \Rightarrow [3.1]$ follows since $[3.2] \Rightarrow [3.3]$ and $[3.3] \Rightarrow [3.1]$. However, we present a direct geometric proof that $[3.2] \Rightarrow [3.1]$ that does not require Dehn’s Lemma or the Loop Theorem. The basic idea is that first integral homology tells us $\lambda$ must be a meridian on $\partial X$. Let $l$ and $m$ denote a longitude and a meridian of $X$. Then $H_1(X) \cong \mathbb{Z}$ is generated by $[l]$. Let $i : \partial \nu K \to X$ denote the inclusion map, so $i : \partial \nu K \to \partial X$ is a homeomorphism. Thus $i(\lambda)$ is an essential simple closed curve in $\partial X$, and $i(\lambda) = am + bl$ for some $a, b \in \mathbb{Z}$ with $\text{gcd}(a, b) = 1$. By the definition of a longitude, $0 = [i(\lambda)] = b[l]$ in $H_1(X)$. Hence $b = 0, a = \pm 1$, and $i(\lambda)$ bounds a disk in $X$. Gluing this disk to an annulus in $\partial \nu K$ bounded by $K$ and $\lambda$ yields a disk in $S^3$ bounded by $K$. Therefore, $K$ is the unknot.

Let $M$ be a connected PL or DIFF 3-manifold (possibly noncompact and possibly with boundary). Recall that $M$ is irreducible provided each embedded $S^2$ in $M$ bounds an embedded $D^3$ in $M$, and $M$ is prime provided each separating, embedded $S^2$ in $M$ bounds an embedded $D^3$ in $M$ (some authors exclude $S^3$). Equivalently, $M$ is prime if and only if $M$ is not a nontrivial connected sum of two 3-manifolds (meaning if $M \approx P \# Q$, then $P \approx S^3$ or $Q \approx S^3$). Recall that $S^1 \times S^2$ is prime, but not irreducible (see Hatcher [Hat07 Prop. 1.4]).

The exterior of every knot in $S^3$ is prime (Alexander’s theorem—that each $S^2$ embedded in $\mathbb{R}^3$ bounds a 3-disk—implies each such exterior is irreducible, and irreducible implies prime). This does not hold for every knot in $S^1 \times S^2$. For instance, consider any knot contained in a 3-disk in $S^1 \times S^2$. The next lemma says that certain knots in $S^1 \times S^2$ do have prime exteriors.
Lemma 3.4. Let $K$ be a knot in $S^1 \times S^2$ with knot exterior $X$. If $H_1(X) \cong \mathbb{Z}$, then $X$ is prime.

Proof. Suppose $X \approx M \# N$. Without loss of generality, assume $\partial M = \partial X \approx T^2$, so $N$ is a closed 3-manifold. It suffices to show that $N \approx S^3$. By the Mayer-Vietoris sequence:

$$\mathbb{Z} \cong H_1(X) \cong H_1(M) \oplus H_1(N)$$

So one of $H_1(M)$ or $H_1(N)$ is trivial and the other is isomorphic to $\mathbb{Z}$. We also have:

$$S^1 \times S^2 = X \cup_{\partial} \nu K \approx (M \# N) \cup_{\partial} \nu K \approx (M \cup_{\partial} \nu K) \# N$$

where $M \cup_{\partial} \nu K$ and $N$ are closed 3-manifolds. As $S^1 \times S^2$ is prime, either $M \cup_{\partial} \nu K \approx S^3$ or $N \approx S^3$, and the other is homeomorphic to $S^1 \times S^2$. If $N \approx S^3$, then we are done. Suppose, by way of contradiction, that $M \cup_{\partial} \nu K \approx S^3$. Thus, $N \approx S^1 \times S^2$, $H_1(N) \cong \mathbb{Z}$, and $H_1(M) = 0$. Lemma 2.10 implies that $\partial M$ is a disjoint union of 2-spheres. This contradicts the fact that $\partial M \approx T^2$ and completes the proof. \qed

Lemma 3.5. Let $\Sigma$ be an essential 2-sphere in $S^1 \times S^2$. Then $\Sigma$ is isotopic to $\{pt\} \times S^2$.

Proof. By general position, we assume $\Sigma$ meets $\{pt\} \times S^2$ transversely in finitely many circles. Let $C$ be such a circle innermost on $\Sigma$. So $C$ bounds a 2-disk $\Delta \subset \Sigma$ such that $\text{Int} \Delta$ is disjoint from $\{pt\} \times S^2$. We also have $\{pt\} \times S^2 = A \cup B$ where $A$ and $B$ are 2-disks and $A \cap B = C$. We claim that one of the 2-spheres $A \cup \Delta$ or $B \cup \Delta$ is inessential in $S^1 \times S^2$.

To see this, observe that $[A \cup \Delta] + [B \cup \Delta] = [A \cup B]$ is the generator of $H_2(S^1 \times S^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$. So exactly one of $[A \cup \Delta]$ or $[B \cup \Delta]$ equals 0 in $H_2(S^1 \times S^2; \mathbb{Z}_2)$. Without loss of generality, assume that $[A \cup \Delta] = 0$, then $A \cup \Delta$ is the boundary of a sum of 3-cells in $S^1 \times S^2$. Denote the sum with $\Gamma$. If we can show that $A \cup \Delta$ is a separating 2-sphere, then we will get that $A \cup \Delta$ is inessential in $S^1 \times S^2$, which is a prime 3-manifold. Suppose, by way of contradiction, that $A \cup \Delta$ does not separate $S^1 \times S^2$. Triangulate $S^1 \times S^2$. Then for each pair of tetrahedra $\sigma$ and $\sigma'$ with $\sigma \cap (A \cup \Delta) = \sigma' \cap (A \cup \Delta)$, one of $\sigma$ or $\sigma'$ is in $\Gamma$ and the other is not. Since $A \cup \Delta$ does not separate, there is a sequence of tetrahedra disjoint from $A \cup \Delta$ that connects $\sigma$ and $\sigma'$. But then either $\sigma$ or $\sigma'$ is in $\Gamma$ would imply that the whole sequence of tetrahedra is in $\Gamma$ for the homology class $[\partial \Gamma] = [A \cup \Delta]$ to be 0. So $\sigma$ and $\sigma'$ are both in $\Gamma$. This contradiction proves the claim.

Now that we have shown either $A \cup \Delta$ or $B \cup \Delta$ is inessential in $S^1 \times S^2$, assume, without loss of generality, that $A \cup \Delta$ bounds a 3-disk in $S^1 \times S^2$. Use this 3-disk to isotope $\Delta$ past $A$, thus eliminating $C$ from $\Sigma \cap \{pt\} \times S^2$ without introducing any
new intersections (circles other than $C$ are also eliminated in case $\Sigma$ meets the interior of $A$). Repeat this process finitely many times so that $\Sigma$ is disjoint from $\{pt\} \times S^2$.

Cut $S^1 \times S^2$ open along $\{pt\} \times S^2$ to get $\Sigma$ contained in the interior of $[0, 1] \times S^2$. It suffices to show there is an isotopy carrying $\Sigma$ to $\{1/2\} \times S^2$. By the 3-dimensional annulus theorem, it suffices to show that $\Sigma$ separates $\{0\} \times S^2$ from $\{1\} \times S^2$ in $[0, 1] \times S^2$. Suppose, by way of contradiction, that there is a simple path from $(0, q)$ to $(1, q)$ for some $q \in S^2$ that is disjoint from $\Sigma$. By the Lightbulb theorem (see Rolfsen [Rol76, p. 257]), we may straighten this path. Thus $\Sigma$ lies in a ball contained in the interior of $[0, 1] \times S^2$ and therefore bounds a ball in $S^1 \times S^2$. This implies $\Sigma$ is inessential in $S^1 \times S^2$, a contradiction. \hfill \Box

**Claim 3.6.** Let $U$ be a knot in $S^1 \times S^2$ of the form $S^1 \times \{pt\}$, and let $K$ be a knot in $S^1 \times S^2$ with knot exterior $X$. Then the following are equivalent.

1. $K$ is isotopic to $U$.
2. $X$ is homeomorphic to $S^1 \times D^2$.
3. $\pi_1(X) \cong \mathbb{Z}$.

**Proof.** The implications $\text{(3.1)} \Rightarrow \text{(3.2)} \Rightarrow \text{(3.3)}$ are straightforward. For the implication $\text{(3.3)} \Rightarrow \text{(3.2)}$, note that the exterior $X$ is compact, connected, orientable, and prime (by Lemma 3.4). Thus $X$ is homeomorphic to $S^1 \times D^2$ by Hatcher [Hat07, Prop. 3.4]. For the implication $\text{(3.2)} \Rightarrow \text{(3.1)}$, the basic idea is that—since we are gluing together two copies of $S^1 \times D^2$ to obtain $S^1 \times S^2$—first integral homology tells us that we must glue a meridian of $\nu K \approx S^1 \times D^2$ to a meridian of $X \approx S^1 \times D^2$. Unlike the case for a knot $K$ in an integral homology 3-sphere, we do not have unique choices for longitudes. Nevertheless, we do have unique choices for meridians, and choices for longitudes are immaterial. Let $\lambda$ and $\mu$ be any longitude and meridian on $\partial \nu K \approx S^1 \times S^1$, so $[\lambda] \text{ generates } H_1(S^1 \times D^2)$ and $\mu$ bounds $\{pt\} \times D^2$. Similarly, let $l$ and $m$ be any longitude and meridian on $\partial X \approx S^1 \times S^1$, so $[l] \text{ generates } H_1(X)$ and $m$ bounds $\{pt\} \times D^2$ in $X \approx S^1 \times D^2$. Gluing gives $[\mu] = [am + bl] = b[l] \in H_1(X) \cong \mathbb{Z}$ generated by $[l]$, where $a, b \in \mathbb{Z}$ and gcd$(a, b) = 1$. As $\mathbb{Z} \cong H_1(S^1 \times S^2) \cong ([l]) / b[l]$, this implies $b = 0$ and $a = \pm 1$, as desired.

Thus we obtain an embedded 2-sphere $\Sigma$ in $S^1 \times S^2$ that is transverse to $K$ and meets $K$ at a single point. Hence, $\Sigma$ is essential in $S^1 \times S^2$, and is therefore isotopic to a 2-sphere of the form $\{q\} \times S^2$ by Lemma 3.5. With $\Sigma$ standard, we apply the Lightbulb theorem (see Rolfsen [Rol76, p. 257]) to $K$ and the result follows. \hfill \Box

**Remark 3.7.** In Claim 3.6, the hypothesis $\pi_1(X) \cong \mathbb{Z}$ cannot be replaced with $H_1(X) \cong \mathbb{Z}$. A pertinent counterexample is Mazur’s knot $K \subset S^1 \times S^2$ in Figure 4.2 ahead. The exterior $X$ of $K$ in $S^1 \times S^2$ is obtained from the link exterior $Y$
of $K \sqcup J$ in $S^3$ by Dehn filling $Y$ so as to accomplish 0-surgery on $J$. As $\pi_1(Y) \cong \langle x_1, x_2, \ldots, x_9 \mid R_1, R_2, \ldots, R_9 \rangle$, we get that $\pi_1(X) \cong \langle x_1, x_2, \ldots, x_9 \mid R_1, R_2, \ldots, R_{10} \rangle$. It is straightforward to abelianize and obtain $H_1(X) \cong \mathbb{Z}$. However, $\pi_1(X) \not\cong \mathbb{Z}$—meaning $X \not\approx S^1 \times D^2$—since a quotient of $\pi_1(X)$ is the infinite, nonabelian $(2, 5, 7)$ triangle group as shown by Mazur (see the next section).

We close this section with two important results on Dehn surgery.

A knot $K \subset S^3$ has Property P provided nontrivial surgery on $K$ never yields a simply-connected 3-manifold. The Property P Conjecture states that all nontrivial knots have Property P.

A knot $K \subset S^3$ has Property R provided surgery on $K$ never yields $S^1 \times S^2$. The Property R Conjecture states that all nontrivial knots have Property R.

The unknot in $S^3$ has neither Property P (since $1/q$-surgery on it yields $S^3$ for each $q \in \mathbb{Z}$) nor Property R (since 0-surgery on it yields $S^1 \times S^2$). Recall that only 0-surgery on a knot $K \subset S^3$ yields an integral homology $S^1 \times S^2$ (see [3.2] above). Progress on both conjectures was incremental since the 1950’s. Bing first proved in 1958 that the trefoil knot has Property P [Bin58, §7]. Further progress was made by González-Acuña [Gon70], Bing and Martin [BM71], Simon [Sim71], and Culler, Gordon, Luecke, and Shalen [CGLS87]. See also Epple [Epp99, §19 & 29] and Rolfsen [Rol76, p. 280]. The full proof of the Property P Conjecture was completed by Kronheimer and Mrowka [KM04] utilizing 4-manifolds and work of several others. Progress on the Property R Conjecture was made by Moser [Mos74], Lambert [Lam77], and Kirby and Melvin [KM78]. The full Property R Conjecture was proved by Gabai [Gab86, Gab87] using foliations.

**Remark 3.8.** Following Laudenbach, a knot in $S^1 \times S^2$ is trivial provided it is isotopic to $S^1 \times \{pt\}$, and otherwise it is nontrivial. For a characterization of trivial knots in $S^1 \times S^2$, see Claim 3.6 above. The Property R Conjecture is equivalent to the claim that Dehn surgery on a nontrivial knot $J \subset S^1 \times S^2$ never yields $S^3$. The proof of this equivalence is straightforward using Claims 3.3 and 3.6.

We will use the following result of Laudenbach [Lau79]. By Remark 3.8, this is a partial result towards the Property R Conjecture.

**Theorem 3.9** (Laudenbach). If $K$ is a nontrivial knot in $S^1 \times S^2$ meeting a sphere $\{pt\} \times S^2$ in at most 3 points, then no surgery on $K$ yields $S^3$, or even a simply-connected 3-manifold.

Laudenbach proved this result using Dehn’s lemma and the Loop theorem. Its proof is much simpler than that of the Property R Conjecture.
4. Mazur Manifolds

Mazur’s 4-manifolds [Maz61] are defined as follows. Let $k$ be an integer, then the Mazur 4-manifold $W_k$ is given by the Kirby diagram in Figure 4.1. It’s a 0-handle union a 1-handle (the dotted circle) union a 2-handle attached along $K$ with framing $k$. The dotted circle notation for 1-handles is due to Akbulut (see [Akb77, p. 100], [AK79, p. 260], and also [GS99, p. 167]).

![Kirby diagram of the Mazur 4-manifold $W_k$ where $K$ is $k$-framed.](image)

Figure 4.1. Kirby diagram of the Mazur 4-manifold $W_k$ where $K$ is $k$-framed.

The boundary 3-manifold $M_k = \partial W_k$ has Kirby diagram as in Figure 4.2. Here, the dotted circle has been replaced with a 0-framed unknot. As the linking matrix of the framed link has determinant $-1$, $M_k$ is a homology 3-sphere. Recall that the framing coefficient of the blackboard framing on $K$ equals the writhe of $K$ (see [GS99, p. 124]); in this case, the blackboard framing is $k = -3$.

Mazur [Maz61] showed that $W_k \times [0, 1] \approx D^5$. In particular, $W_k$ is contractible. Mazur also showed that $M_{-3}$ is not simply-connected (we include Mazur’s argument below). In particular, $W_{-3}$ is a compact, contractible 4-manifold distinct from $D^4$.

Two natural problems arise that we will address:

(4.1) Show every $M_k$ is not simply-connected.

(4.2) Show the $M_k$’s have pairwise nonisomorphic fundamental groups.

In particular, affirmative answers would imply that the $W_k$’s are pairwise non-homeomorphic compact, contractible 4-manifolds not homeomorphic to $D^4$. 
We compute the fundamental group of $M_k$. The generators $x_1, x_2, \ldots, x_9$ are shown.
in Figure 4.3 The relations are as follows.

\[
\begin{align*}
R_1 \quad & x_1x_5 = x_6x_1 & R_7 \quad & x_4x_8 = x_7x_4 \\
R_2 \quad & x_2x_4 = x_5x_2 & R_8 \quad & x_1x_8 = x_9x_1 \\
R_3 \quad & x_4x_2 = x_3x_4 & R_9 \quad & x_6x_9 = x_7x_6 \\
R_4 \quad & x_7x_6 = x_1x_7 & R_{10} \quad & 1 = x_4x_1^{-1}x_6^{-1} \\
R_5 \quad & x_7x_1 = x_2x_7 & R_{11} \quad & 1 = x_7^{-1}x_4^{-1}x_7x_2^{-1}x_1^{-1}x_7^{-1}x_1^{k+3} \\
R_6 \quad & x_7x_4 = x_3x_7
\end{align*}
\]

Relations \( R_1, R_2, \ldots, R_9 \) are the Wirtinger relations corresponding to the crossings in the diagram. The link complement \( S^3 - (K \sqcup J) \) has fundamental group presented by \( \langle x_1, x_2, \ldots, x_9 \mid R_1, R_2, \ldots, R_9 \rangle \). The relations \( R_{10} \) and \( R_{11} \) correspond to the surgeries on \( J \) and \( K \), respectively. Note that the blackboard framing on \( K \) corresponds to \( k = -3 \). The fundamental group of \( M_k \) is presented by \( \langle x_1, x_2, \ldots, x_9 \mid R_1, R_2, \ldots, R_{11} \rangle \).

We simplify this presentation with Tietze transformations. Use \( R_4 \) to eliminate \( x_6 \), use \( R_5 \) to eliminate \( x_5 \), use \( R_5 \) to eliminate \( x_2 \), use \( R_{10} \) to eliminate \( x_4 \), use \( R_6 \) to eliminate \( x_3 \), use \( R_7 \) to eliminate \( x_8 \), and use \( R_9 \) to eliminate \( x_9 \). Note that \( R_8 \) collapses to 1 = 1, and \( R_3 \) is a cyclic permutation of \( R_2 \). We now have the presentation:

\[
\pi_1(M_k) \cong \langle x_1, x_7 \mid x_7^{-2}x_1 x_7 x_1 x_7^{-1}x_1^{-1}x_1^{-1}x_7 x_1 x_7 x_1 = 1, x_7^{-1}x_1^{-1}x_7^{-1}x_1^{-1}x_7^{-1}x_1^{-1}x_7^{-1}x_1^{k+3} = 1 \rangle
\]

Following Mazur [Maz61] p. 227 we make the substitutions (which are readily accomplished with Tietze transformations): \( x_1 = b^{-1}a \) and \( x_7 = b \). The two relations become: \( b^{-3}a^2b^{-3}a^2 = 1 \) and \( b^{-1}a^{-2}b^4a^{-2}(b^{-1}a)^{k+3} = 1 \). This gives:

\[
\pi_1(M_k) \cong \langle a, b \mid b^{-3}a^2b^{-3}a^2 = 1, b^{-1}a^{-2}b^4a^{-2}(b^{-1}a)^{k+3} = 1 \rangle
\]

The exponent sum matrix of this presentation equals \[
\begin{pmatrix}
1 & k-1 \\
-1 & -k
\end{pmatrix}
\]

which is unimodular with determinant -1. Therefore, this group is perfect for each integer \( k \).

For the specific case \( k = -3 \), the second relation is: \( b^4 = a^2ba^2 \). Using this, we show that the first relation becomes \( a^7 = b^5 \) as noted by Mazur [Maz61] p. 227: \( b^{-3}a^2b^2a^{-2}a^{-3}a^{-2}a^2b^2 = 1 \iff b^{-3}b^4a^{-7}b^4 = 1 \iff a^{-7} = b^{-5} \). We have thus obtained Mazur’s presentation \( \pi_1(M_{-3}) \cong \langle a, b \mid b^5 = a^7, b^4 = a^2ba^2 \rangle \). Include the relation \( b^5 = 1 \) to obtain the quotient \( \langle a, b \mid a^7, b^5, (ba^2)^2 \rangle \). Include the generator \( c \) and the relation \( c = a^2 \). A few Tietze transformations yields \( \langle b, c \mid b^5, c^7, (bc)^2 \rangle \) which presents the (2, 5, 7) hyperbolic triangle group. As this quotient of \( \pi_1(M_{-3}) \) is infinite and nonabelian, \( \pi_1(M_{-3}) \) is infinite and nonabelian as well. In particular, \( M_{-3} \)
is not homeomorphic to $S^3$, and $W_{-3}$ is not homeomorphic to $D^4$.

The remainder of this section is devoted to proving the next theorem.

**Theorem 4.1.** Each of the groups $\pi_1(M_k)$, $k \in \mathbb{Z}$, is nontrivial. For $-200 \leq k \leq 200$, the groups $\pi_1(M_k)$ are pairwise nonisomorphic.

The earliest proof that $\pi_1(M_k)$ is never trivial appears to be due to Laudenbach sometime prior to 1979 [Lau79]. Akbulut and Kirby [AK79, p. 263] state that Laudenbach and Eaton both proved this result, though we have been unable to locate Eaton’s proof. Unfortunately, since that time, Eaton has passed. To prove this result, note that, by Laudenbach’s Theorem 3.9, it suffices to show that Mazur’s knot $K \subset S^1 \times S^2$ in Figure 4.2 is nontrivial. This follows from Remark 3.7 above. Alternatively, as $K \subset S^1 \times S^2$ is nontrivial, the Property R Conjecture shows that no $M_k$ is $S^3$ by Remark 3.8 hence no $M_k$ is simply-connected by the 3-dimensional Poincaré Conjecture. This second approach—mentioned to us by both Akbulut and Laudenbach—uses much more machinery than Laudenbach’s first approach. For more on Property R and conjectures of Laudenbach and Poénaru, see pages 17–19 of Kirby’s *Problems in Low-Dimensional Topology*.

The Casson invariant also shows that no $M_k$ is simply-connected. Using Theorem 6.3 of Matveev and Polyak [MP09, pp. 849–850], we compute Casson’s (original) invariant $\lambda(M_k) = -2$ for each $k \in \mathbb{Z}$. (Note that Matveev and Polyak compute the Casson-Walker invariant which is twice Casson’s original invariant for integral homology 3-spheres.) This implies that each $M_k$ is not simply-connected [AM90, p. xiv].

Here is a completely algebraic proof that each $\pi_1(M_k)$ is nontrivial. Include the extra relation $(b^{-1}a)^{12} = 1$ to obtain the quotient of $\pi_1(M_k)$:

$$G(k) = \left\langle a, b \mid b^{-3}a^2ba^{-3}ba^2 = 1, b^{-1}a^{-2}b^4a^{-2}(b^{-1}a)^{k+3} = 1, (b^{-1}a)^{12} = 1 \right\rangle$$

The isomorphism type of $G(k)$ is determined by $k \mod 12$. For $0 \leq k \leq 11$, $G(k)$ has a nontrivial simple group as a quotient. We find these using MAGMA’s SimpleQuotients command. This yields concrete surjective homomorphisms from $G(k)$ to permutation groups.

For the second claim in the theorem, we used MAGMA to compute the abelianizations of the low index subgroups (index $\leq 13$) of $\pi_1(M_k)$ for $-200 \leq k \leq 200$. This
calculation took 8.3 hours CPU time. The numbers of these low index subgroups alone don’t distinguish these fundamental groups, their abelianizations are necessary.

**Conjecture 4.2.** The fundamental groups $\pi_1(M_k)$ are pairwise nonisomorphic for all $k \in \mathbb{Z}$.

5. **Jester Manifolds**

Jester 4-manifolds were defined by Sparks [Spa18, p. 2137] as follows. Let $k$ be an integer, then the Jester 4-manifold $J_k$ is given by the Kirby diagram in Figure 5.1 (We use the definition in Sparks’ Figure 3, which seems to differ from the sentence prior to that figure.)

![Figure 5.1. Kirby diagram of the Jester manifold $J_k$ where $K$ is $k$-framed.](image)

The boundary 3-manifold $\partial J_k$ has Kirby diagram as in Figure 5.2. As the linking matrix of the framed link has determinant $-1$, $\partial J_k$ is a homology 3-sphere. Here the blackboard framing is $k = 4$.

Sparks [Spa18, p. 2138] asked the following questions:
(5.1) Does there exist $J_k$ not homeomorphic to $D^4$?
(5.2) Do there exist infinitely many homeomorphism types among the $J_k$’s?
We address these questions in Theorem 5.1 below.

We compute the fundamental group of $\partial J_k$. The generators $x_1, x_2, \ldots, x_9$ are
Figure 5.2. Kirby diagram of the boundary $\partial J_k$ of the Jester manifold $J_k$ where $K$ is $k$-framed and $J$ is 0-framed.

Figure 5.3. Generators of the fundamental group of $\partial J_k$.

shown in Figure 5.3. The relations are as follows.

\[
\begin{align*}
R_1 & \ x_1x_3 = x_2x_1 & \quad & R_7 & \ x_8x_1 = x_2x_8 \\
R_2 & \ x_5x_4 = x_3x_5 & \quad & R_8 & \ x_4x_8 = x_10x_4 \\
R_3 & \ x_3x_6 = x_5x_3 & \quad & R_9 & \ x_7x_9 = x_10x_7 \\
R_4 & \ x_3x_1 = x_7x_3 & \quad & R_{10} & \ x_1x_9 = x_8x_1 \\
R_5 & \ x_8x_4 = x_5x_8 & \quad & R_{11} & \ 1 = x_4^{-1}x_7x_1^{-1} \\
R_6 & \ x_8x_7 = x_6x_8 & \quad & R_{12} & \ 1 = x_1x_5x_8^{-1}x_3x_8x_3x_8^{-1}x_2^{-k-4}
\end{align*}
\]
Relations $R_1, R_2, \ldots, R_{10}$ are the Wirtinger relations corresponding to the crossings in the diagram. The link complement $S^3 - (K \cup J)$ has fundamental group presented by $\langle x_1, x_2, \ldots, x_{10} \mid R_1, R_2, \ldots, R_{10} \rangle$. The relations $R_{11}$ and $R_{12}$ correspond to the surgeries on $J$ and $K$, respectively. Note that the blackboard framing on $K$ corresponds to $k = 4$. The fundamental group of $\partial J_k$ is presented by $\langle x_1, x_2, \ldots, x_{10} \mid R_1, R_2, \ldots, R_{12} \rangle$. We simplify this presentation with Tietze transformations. Use $R_7$ to eliminate $x_1$, use $R_1$ to eliminate $x_3$, use $R_{10}$ to eliminate $x_9$, use $R_4$ to eliminate $x_7$, use $R_6$ to eliminate $x_6$, use $R_9$ to eliminate $x_{10}$, use $R_{11}$ to eliminate $x_4$, and use $R_5$ to eliminate $x_5$. Note that $R_4$ and $R_8$ collapse to 1 = 1, and $R_3$ is a cyclic permutation of $R_2$. We now have the following presentation of $\pi_1(\partial J_k)$ with two relations:

$$
\begin{align*}
&\begin{cases}
x_2^{-1}x_8x_2x_8^{-1}x_2x_8x_2^{-1}x_2^{-1}x_2^{-1}x_8x_2x_8^{-1}x_2x_8x_2^{-1}x_2^{-1}x_2^{-1}x_8x_2x_8^{-1}x_2^{-1}x_2^{-1}
x_2, x_8
\end{cases} \\
x_2^{-1}x_8x_2x_8^{-1}x_2x_8x_2^{-1}x_2^{-1}x_2^{-1}x_8x_2x_8^{-1}x_2^{-1}x_2^{-1}x_8x_2x_8^{-1}x_2^{-1}x_2^{-1}
\end{align*}
$$

We make the following substitutions: $x_2 = a$ and $x_2x_8^{-1} = b$. The two relations become: $abab^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}bab^{-2} = 1$ and $bab^{-2}abab^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1} = 1$. This gives:

$$
\pi_1(\partial J_k) \cong \begin{cases}
a, b & abab^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}bab^{-2} = 1, \\
& bab^{-2}abab^{-1}a^{-1}ba^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1} = 1
\end{cases}
$$

The exponent sum matrix of this presentation equals $\begin{bmatrix} -1 & k - 2 \\ 0 & 1 \end{bmatrix}$, which is unimodular with determinant $-1$. Therefore, this group is perfect for each integer $k$.

The remainder of this section is devoted to proving the next theorem.

**Theorem 5.1.** Each of the groups $\pi_1(\partial J_k)$, $k \in \mathbb{Z}$, is nontrivial. For $-100 \leq k \leq 100$, the groups $\pi_1(\partial J_k)$ are pairwise nonisomorphic.

As in the proof of Theorem 4.1 above, Lazard’s Theorem 3.9 shows $\pi_1(\partial J_k)$ is never trivial provided we know that the knot $K \subset S^1 \times S^2$ in Figure 5.2 is nontrivial. Let $X$ be the exterior of $K$ in $S^1 \times S^2$. So $\pi_1(X) \cong \langle x_1, x_2, \ldots, x_{10} \mid R_1, R_2, \ldots, R_{11} \rangle$. MAGMA shows that $\pi_1(X)$ surjects to $A_5$, the alternating group of order 60. This proves the first claim in Theorem 5.1 and answers Sparks’ first question in the affirmative for every $J_k$.

The Casson invariant of each $\partial J_k$ turns out to be 0. We again used Matveev and Polyak [MP99] to perform this calculation.
Here is a completely algebraic proof that each \( \pi_1(\partial J_k) \) is nontrivial. Include the extra relation \( a^{12} = 1 \) to obtain the quotient of \( \pi_1(\partial J_k) \):

\[
H(k) = \left\{ \begin{array}{l}
abab^{-1}a^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1}b^{-1}a^{-1}b^2a^{-1}b^{-1}a^{-1}bab^{-2} = 1, \\
abab^{-2}aba^{-1}ba^{-1}b^{-1}abab^{-2}aba^{-1}b^{-1}abab^{-2}aba^{-1}b^2a^{-1}b^{-1}a^{-1}bab^{-2} = 1, \\
a^{12} = 1
\end{array} \right.
\]

The isomorphism type of \( H(k) \) is determined by \( k \mod 12 \). For \( 0 \leq k \leq 11 \), \( H(k) \) has a nontrivial simple group as a quotient. We find these using MAGMA’s Simple-Quotients command. This yields concrete surjective homomorphisms from \( H(k) \) to permutation groups. This completes our algebraic proof of the first claim in Theorem 5.1.

For the second claim in the theorem, we used MAGMA to compute the abelianizations of the low index subgroups (index \( \leq 11 \)) of \( \pi_1(\partial J_k) \), \( -100 \leq k \leq 100 \). This calculation took 5 hours CPU time.

**Conjecture 5.2.** The fundamental groups \( \pi_1(\partial J_k) \) are pairwise nonisomorphic for all \( k \in \mathbb{Z} \).

### 6. Future Directions

We close this thesis with some problems for further study. It appears that Sparks defined two types of Jester manifolds [Spa18, p. 2137]. These two definitions should be compared closely and the resulting 4-manifolds, together with the Mazur manifolds, should all be compared to one another. Further invariants of these manifolds should be computed: the Casson invariant of each Jester manifold, the Floer homology of all of these boundaries, and the contact invariant of the boundaries of the Stein 4-manifolds (the latter suggested to us by Akbulut).

Another possible approach to proving the boundary of each Mazur manifold is nonsimply-connected raises an intriguing question about triangle groups. Begin with our presentation of \( \pi_1(M_k) \) on the generators \( a \) and \( b \) in Section 4. Add the relations \( a^7 = 1 \) and \( b^5 = 1 \). Add the generator \( c \) and the relation \( c = a^2 \) to obtain the following quotient of \( \pi_1(M_k) \): \( L(k) = b, c \mid b^5, c^7, (bc)^2, (bc^{3^{k+3}}). \) This is also a quotient of the hyperbolic triangle group \( \Delta(2, 5, 7) \). It can be shown that the element \( bc^{3^{k+3}} \) has infinite order in \( \Delta(2, 5, 7) \). When \( k + 3 = 0 \), \( L(k) \cong \Delta(2, 5, 7) \) is infinite and nonabelian. When \( k + 3 \in \{1, 2, 3\} \), \( L(k) \) is the trivial group. We conjecture that \( L(k) \) is nontrivial for all \( k + 3 \geq 4 \). We verified this conjecture for \( 4 \leq k \leq 30 \). This conjecture would give a quick, uniform algebraic proof that all but six Mazur manifold boundaries are nonsimply-connected. We have at least two other such quotients
of triangle groups that would accomplish the same result.

The general question arises: given a hyperbolic triangle group \( \Delta \) and an element \( w \in \Delta \) of infinite order, for which positive integers \( k \) is the normal closure of \( w^k \) a proper subgroup of \( \Delta \)? It seems interesting to determine such \( k \) based on the geometry of \( \Delta \).

References


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