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Proper 3-colorings of cycles and hypercubes

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Honors paper
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1 Introduction

In this paper, we look at two families of graphs, cycles and hypercubes, and compare how their sets of proper 3-colorings differ as the graphs get arbitrarily large. (See Section 2 for definitions.) In particular, we find the probability of pairs of vertices at various distances being the same color in order to understand the range and scale of interactions between them. As we look at larger and larger cycles, larger and larger hypercubes, patterns begin to emerge.

If we pick a proper 3-coloring of C_n, the cycle with n vertices, uniformly randomly out of the set of all possible 3-colorings of C_n, as n goes to infinity, there will be about the same number of vertices of each color. The probability that two vertices a fixed fraction of the cycle away from each other will be the same color is $\frac{1}{3}$, what we would expect if they were colored independently.

Interestingly, the same is not true for the hypercube. Note that Q_n is bipartite, so we can call half of the vertices the "even" vertices and the other half the "odd" vertices. In fact, as n gets arbitrarily large, a uniformly randomly chosen proper 3-coloring of Q_n will have almost all of the even vertices one color and almost all of the odd vertices a random smattering of the other two colors, or vice versa. So, we actually get that any odd vertex has $\frac{3}{4}$ probability of being the same color as another odd vertex, the same for even, and that the probability of an odd and even vertex being the same color is actually 0 as n goes to infinity. This means that typical proper 3-colorings of arbitrarily large hypercubes are "basically just 2-colorings" in the sense that, ignoring the small number of vertices that deviate from the pattern, identifying the two colors that share a parity would result in a 2-coloring of the graph.

As for how we go about proving these statements, cycles are relatively straightforward. We will use standard methods from probability and combinatorics to count the number of proper 3-colorings with certain characteristics, and then just divide by the total number of proper 3-colorings. Hypercubes, however, in no way lend themselves to that kind of direct enumeration. Instead, for hypercubes we use methods developed in [5] by Jenssen and Keevash, which takes a statistical physics approach to homomorphisms from the discrete torus. Their results then apply to proper 3-colorings of the hypercube since a hypercube is a simpler case of the discrete torus and a proper 3-coloring is a type of graph homomorphism. One of the essential tools they provide is a way to understand the number of *defect vertices*, the vertices that do not fit with the pattern of half of the hypercube being one color, and the other half being the other two colors.

These results are interesting for many reasons. Proper colorings of graphs have been used to solve many different problems, from assigning radio frequencies to cell phone towers so that they don't overlap, to assigning colors to regions on a map so that adjacent regions are different colors, to many more sophisticated problems in physics, engineering, etc. Graphs model networks, and proper colorings are often used to model various properties of such networks. Not only are proper colorings interesting and useful, but they are also difficult to understand. In fact, determining whether a graph has a proper k-coloring is NP-complete for $k \geq 3$ [3]. The simplest nontrivial case is when $k = 3$, and we can understand the collection of proper 3-colorings of the cycle and hypercube, albeit through different methods. The fact that proper 3-colorings behave so differently on these two families of graphs gives an example of how local constraints can propagate radically differently through different types of networks.

The paper is organized as follows. In Section 2, we give graph theory and proper coloring background. In Section 3, we focus on cycles, while we focus on hypercubes in Section 4. In Section 5, we compare our results and conclude the paper.

2 Background

To understand proper 3-colorings of cycles and hypercubes, we need some graph theory background first.

A *graph* G consists of a set of *vertices* V and a set of ordered pairs of vertices called *edges* E. Note that $V(G)$ and $E(G)$ refer to the vertex and edge sets of G, respectively. A graph G is connected if there is a path of edges from each vertex to every other vertex in G, and the *distance* between a vertex u and v in G is the minimum number of edges that form a path between them. Two vertices are *adjacent* if they share an edge, every vertex that shares an edge with u is called a *neighbor* of u, and the *degree* of a vertex is the size of the set of its neighbors.

Now, we define our main two graph families. A *cycle* is a connected graph where every vertex has degree exactly two. As shown in Figure 1, we let C_n denote the cycle with n vertices. Additionally, a *path* with n vertices is denoted P_n and is just C_n but with one edge removed.

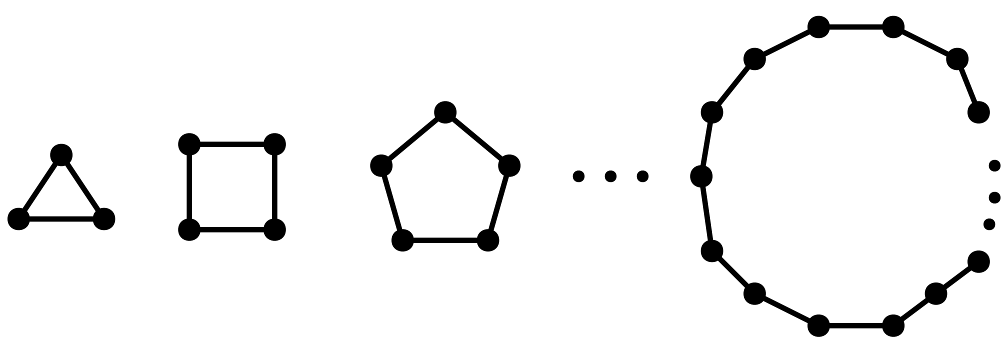

Figure 1: The cycless C_3, C_4, and C_5 are shown along with an arbitrarily large cycle.

A hypercube, on the other hand, is much more complicated. Let Q_n denote the *hypercube* of dimension n. Q_0 is defined as an isolated vertex, and then for each $n > 0$, Q_n consists of two copies of Q_{n-1} where corresponding vertices are connected by an edge. As shown in Figure 2, Q_0 is an isolated vertex, Q_1 is a single edge, Q_2 is the square, Q_3 is the cube, and Q_n is the n-dimensional generalization of a cube. Note that Q_n has 2^n vertices.

Additionally, we can partition the set of vertices of Q_n into two sets of vertices $\mathcal{E}$ and $\mathcal{O}$ such that all edges in Q_n go from one set to the other. This can just be done by picking a random vertex v and putting it in $\mathcal{E}$, putting its neighbors in $\mathcal{O}$, putting their neighbors in

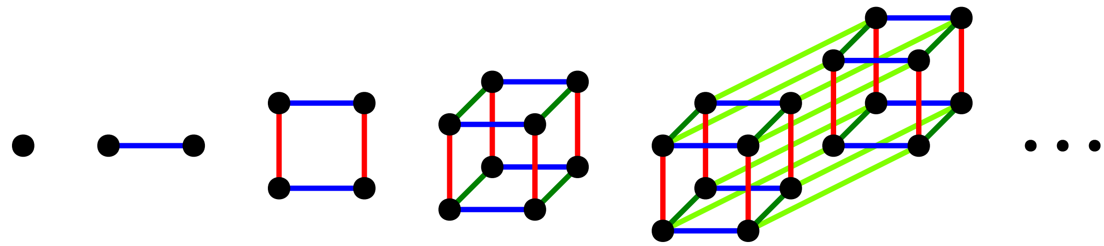

Figure 2: The hypercubes $Q_0, Q_1, \ldots, Q_4$ are shown.

$\mathcal{E}$, etc. Thus, Q_n is called *bipartite* and we call $\mathcal{E}$ the set of *even vertices* and $\mathcal{O}$ the set of *odd vertices*.

Now, we define a *proper k-coloring* of a graph. A *k-coloring* of a graph G assigns one of k colors to each vertex in G, while a *proper k-coloring* of G is a k-coloring of G where no adjacent vertices are the same color. In this paper, we consider proper 3-colorings specifically, and assign either red, blue, or green to the vertices of a cycle or hypercube such that there is no edge going between two vertices of the same color.

Note that C_n is 2-colorable if and only if n is even. If vertex v is colored red, the vertex to the right of it must be blue, the next is be red, the next blue, and so on; if n is even, then we end the process by coloring the vertex to the left of v blue, and it is a proper 2-coloring, but n is odd, then we end the process by coloring the vertex to the left of v red too, so it is not a proper 2-coloring. Since we are comparing proper 3-colorings of hypercubes to proper 2-colorings, this motivates our choice to only consider the even cycle later on.

In fact, all cycles and hypercubes are 3-colorable. The hypercube Q_n is bipartite, so is 2-colorable since we can just color the even vertices one color and the odd vertices the other. Since we can color Q_n with two colors, we can trivially color it with three. As for C_n, the cycle on n vertices, we already know from above that the even cycle is 2-colorable, hence 3-colorable, and use a similar argument for the odd cycle. Given an odd cycle with $2k + 1$ vertices, we can alternate between two colors for the first $2k$ vertices, and then color vertex $2k+1$ the third color. So, for the rest of the paper, we will never have to be concerned about whether a proper 3-coloring exists, just what properties the collection of them have.

Additionally, it is important to note that, for a uniformly randomly chosen proper 3-coloring of a fixed graph G, the probability that any single vertex v is any particular color is $\frac{1}{3}$ by symmetry. The are exactly as many ways to color G with v colored red, blue, or green since, to get to a coloring with v colored red to a coloring with v colored green, we can just permute the colors (for example, recolor all red vertices as green, green as blue, and blue as red). Correspondingly, if the colors of two vertices u and v in some graph G are independent, then that means that

$$P(u \text{ and } v \text{ are the same color}) = 3\left(\frac{1}{3^2}\right) = \frac{1}{3}.$$

Later, we will find that the colors of vertices at distance ϵn, for any $\epsilon > 0$, in the cycle are indeed independent as n goes to infinity, though the same is not true for the hypercube.

3 3-colorings of cycles

In this section, we investigate what a uniformly randomly selected proper 3-coloring of an even cycle C_{2k} will generally look like. How likely is it that two vertices at distance two from each other are the same color? That vertices on opposite sides of the cycle are the same color? That one third of the vertices are red? These are some of the questions that we will answer in this section.

3.1 The chromatic polynomial for cycles

First, we need to find the total number of proper 3-colorings of the cycle. Later, this will allow us to directly count the number of 3-colorings with certain properties and divide by the total number of possible colorings to obtain the probabilities that we are interested in.

To find the total number of colorings, we make use of the chromatic polynomial for C_n. While the two results that we will prove related to the chromatic polynomial are not new (see [1, §V]), their proofs provide insight as to why such a simple chromatic polynomial does not exist for hypercubes.

The *chromatic polynomial* $P_G(x)$ for some graph G is a function from the positive integers to the positive integers where $P_G(x)$ represents the number of ways to properly x-color G. To find this polynomial for C_n, we first prove the deletion-contraction recurrence for chromatic polynomials. In the statement of the recurrence and as shown in Figure 3, $\setminus$ denotes *deletion* of an edge e (e is removed from the set of edges of C_k) and / denotes *contraction* of an edge e (again, $e = (u, v)$ is deleted, but the vertices u and v are also identified).

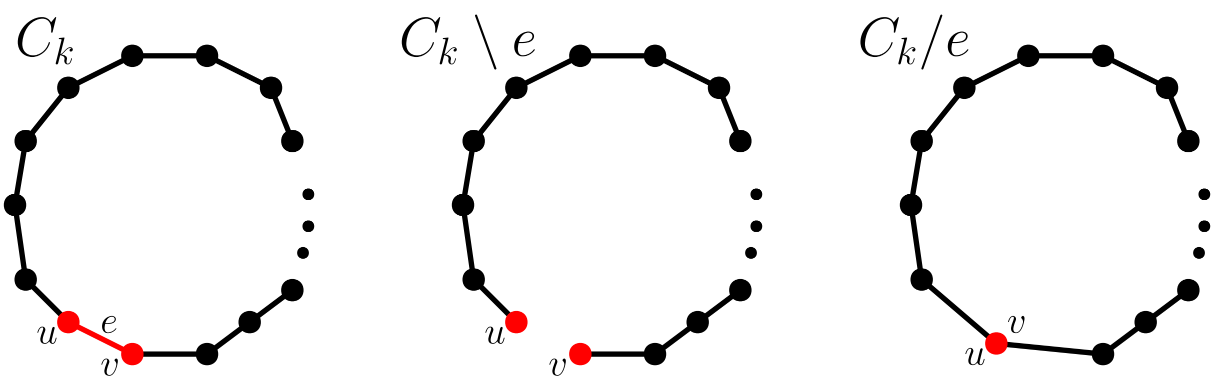

Figure 3: C_k, $C_k \setminus e$, and C_k/e are shown.

Lemma 3.1. *For any edge e in the cycle C_k,*

$$P_{C_k}(x) = P_{C_k \setminus e}(x) - P_{C_k/e}(x).$$

Proof. If $e = (u, v)$, then $P_{C_k \setminus e}(x)$ represents the number of colorings of C_k that are proper colorings in all ways except that u and v may be the same color. To count the number of ways that we can color C_k as a proper coloring except that u and v are the same color, we contract e and count the number of colorings for C_k/e, $P_{C_k/e}(x)$. This way, we have identified u and v, so they must be the same color, while the rest of the coloring is proper. Thus, if we subtract $P_{C_k/e}(x)$ from $P_{C_k\backslash e}(x)$, we get the total number of proper colorings (where u and v are different colors).

□

We can now use this lemma to compute exactly the number of proper 3-colorings of any cycle.

Proposition 3.2. *The chromatic polynomial for C_k is*

$$P_{C_k}(x) = (x-1)^k + (-1)^k(x-1).$$

Proof. We use induction on k.

Base Case: When $k = 3$, label the three vertices of C_3 as v, u, and w. Then, there are x ways to color v, then $x - 1$ ways to color u since it cannot be the same color as v, and finally $x - 2$ ways to color w since it has to be different from both u and v. This gives us that there are

$$P_{C_3}(x) = x(x-1)(x-2)$$

total x-colorings of C_3.

Using the formula given above, we get that

$$\begin{aligned} P_{C_3}(x) &= (x-1)^3 + (-1)^3(x-1) \\ &= (x-1)^3 - (x-1) \\ &= (x-1)\left((x-1)^2 - 1\right) \\ &= x(x-1)(x-2), \end{aligned}$$

which is correct!

Inductive Step: Now, assume that

$$P_{C_{k-1}}(x) = (x-1)^{k-1} + (-1)^{k-1}(x-1),$$

for some $k > 3$ and consider P_{C_k}. By the deletion-contraction recurrence and Figure 3, we know that

$$\begin{aligned} P_{C_k}(x) &= P_{C_k\backslash e}(x) - P_{C_k/e}(x) \\ &= P_{P_k}(x) - P_{C_{k-1}}(x) \end{aligned}$$

where P_k is the path on k vertices. There are $x(x-1)^{k-1}$ ways to properly color P_k with x colors since we can color the first vertex one of x colors, and then each vertex after that can be any of the x colors except the color of the vertex preceding it. Thus, we can substitute in for $P_{P_k}(x)$ and $P_{C_{k-1}}(x)$ from our inductive step to get that

$$\begin{aligned}
P_{C_k}(x) &= P_{P_k}(x) - P_{C_{k-1}}(x) \\
&= x(x-1)^{k-1} - ((x-1)^{k-1} + (-1)^{k-1}(x-1)) \\
&= x(x-1)^{k-1} - (x-1)^{k-1} + (-1)^{k}(x-1) \\
&= (x-1)(x-1)^{k-1} + (-1)^{k}(x-1) \\
&= (x-1)^{k} + (-1)^{k}(x-1),
\end{aligned}$$

which is exactly what we wanted!

□

Notice that, while the deletion-contraction recurrence is simple for cycles since it gives us a way to understand the chromatic polynomial for a cycle in terms of a path and another cycle, the deletion-contraction recurrence is usually not as helpful. For example, if we delete or contract an edge of a hypercube, we are left with two even less well understood types of graphs. So, while this strategy works great for cycles, it is one of the many strategies that just becomes too complicated to be useful for hypercubes.

To give an example of how this polynomial works for cycles, we compute

$$\begin{aligned}
P_{C_k}(2) &= (2-1)^{k} + (-1)^{k}(2-1) \\
&= 1 + (-1)^{k} \\
&= \begin{cases} 2 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

This means that there are two ways to properly 2-color a cycle with an even number of vertices, but zero ways to properly 3-color a cycle with an odd number of vertices. As we saw in Section 2, this is correct.

Additionally, since we will focus on even cycles and proper 3-colorings going forward, we can simplify this result to just be

$$\begin{aligned}
P_{C_{2m}}(3) &= (3-1)^{2m} + (-1)^{2m}(3-1) \\
&= 2^{2m} + 2,
\end{aligned}$$

so there are $2^{2m} + 2$ proper 3-colorings of a cycle with $2m$ vertices.

Now that we know the total number of proper 3-colorings of any cycle C_{2m}, we can answer the questions related to what a uniformly randomly selected proper 3-coloring of C_{2m} will generally look like.

3.2 Probabilities

In this section, we compute specific probabilities that will give us a concrete understanding of the structure of proper 3-colorings in arbitrarily large even cycles. While we will show that the colors of vertices at fixed distances from each other are not independent, the colors of vertices a fixed fraction of the cycle away from each other are. As we might expect, a uniformly randomly chosen proper 3-coloring of the cycle will have about as many vertices of each color.

First, we consider the probability that some fixed vertex u is the same color as another vertex v at some fixed distance away from u. If we label one vertex v_1 and then label the rest of the vertices in order $v_2, v_3, \ldots, v_{2m}$, then we want to understand

$$f(k) := \lim_{m\to\infty} P(v_1 \text{ and } v_k \text{ are the same color}).$$

It is clearly true that $f(1) = 1$ since v_1 is the same color as itself and $f(2) = 0$ since v_2 is v_1's neighbor. Additionally, we know from the previous section that there are $P_{C_{2m}}(3) = 2^{2m} + 2$ total proper 3-colorings of C_{2m}.

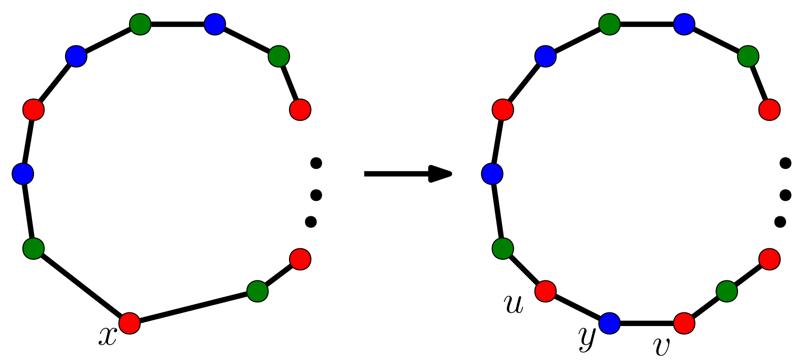

Figure 4: We properly 3-color C_{2m-2}, then replace vertex x with vertices u, y, and v to turn it into a proper 3-coloring of C_{2m} where u and v are the same color.

To find $f(3)$, given fixed $u, v \in V(C_{2m})$ at distance two from each other, we need to count the number of proper 3-colorings such that u and v are the same color. To do so, we first properly 3-color C_{2m-2}, and there are $P_{C_{2m-2}}(3)$ ways to to do this. Then, we take a specific vertex x in C_{2m-2} and replace it with u, v, and a vertex y between u and v. We keep u and v the same color as x, and then have two options for the color of y. As seen in Figure 4, the coloring of this new cycle is a proper 3-coloring of C_{2m} where u and v are the same color, and in fact every proper 3-coloring where u and v are the same color can be found in this way. This means that we can find all colorings with u and v the same color by finding the number of colorings of C_{2m-2}, and then multiplying by two to decide the color of y. Thus,

$$\begin{aligned} f(3) &= \lim_{m\to\infty} \frac{2 \cdot P_{C_{2m-2}}(3)}{P_{C_{2m}}(3)} \\ &= \lim_{m\to\infty} \frac{2(2^{2m-2}+2)}{2^{2m}+2} \\ &= \lim_{m\to\infty} \frac{2^{2m-1}+4)}{2(2^{2m-1}+1)} = \frac{1}{2}. \end{aligned}$$

Therefore, for arbitrarily large m, a given pair of vertices $u, v \in V(C_{2m})$ at distance two have a $\frac{1}{2}$ chance of being the same color. Thus, they are not asymptotically independent.

Now, we consider all integers $k > 3$ and vertices u and v at distance k. Similar to above, we can split C_{2m} into two cycles that each have one vertex that represents both u and v: the cycle with $k-1$ vertices that will tell us how to color the vertices between u and v, and the cycle with $2m-k+1$ vertices that will tell us how to color the rest of the cycle. We then divide by three so that the vertices in either cycle that correspond to u and v are the same color, and then we have a coloring of C_{2m} where u and v are the same color. Again, any such coloring of C_{2m} can be found through this process. Using this method, we compute

$$\begin{aligned} f(k) &= \lim_{m\to\infty} \frac{P_{C_{k-1}} P_{C_{2m-k+1}}}{3(2^{2m}+2)} \\ &= \lim_{m\to\infty} \frac{\left(2^{k-1}+(-1)^{k-1}2\right)\left(2^{2m-k+1}+(-1)^{2m-k+1}2\right)}{3\cdot 2^{2m}+6} \\ &= \lim_{m\to\infty} \frac{2^{2m}+(-1)^{k-1}2^{2m-k+2}+(-1)^{2m-k+1}2^{k}+4}{3\cdot 2^{2m}+6} \\ &= \lim_{m\to\infty} \frac{1+(-1)^{k-1}2^{-k+2}+(-1)^{2m-k+1}2^{k-2m}+2^{2-2m}}{3+3\cdot 2^{1-2m}} \\ &= \frac{\lim_{m\to\infty}\left(1+(-1)^{k-1}2^{-k+2}+(-1)^{2m-k+1}2^{k-2m}+2^{2-2m}\right)}{\lim_{m\to\infty}\left(3+3\cdot 2^{1-2m}\right)} \\ &= \frac{1+(-1)^{k-1}2^{-k+2}}{3}. \end{aligned}$$

Thus, the colors of v_1 and a fixed vertex v_k are not asymptotically independent as the rest of the cycle increases in size. Here is table of the first couple results,

k	$f(k)$
1	1
2	0
3	$\frac{1}{2}$
4	$\frac{1}{4}$
5	$\frac{3}{8}$
6	$\frac{5}{16}$
7	$\frac{11}{32}$
$\vdots$	$\vdots$

and a visual representation is shown in Figure 5. Each higher value of k switches whether $f(k)$ is above or below $\frac{1}{3}$, but each is closer than the last. In a sense, this is the effect of the bipartite-ness of the graph slowly becoming less relevant as the vertices get farther away.

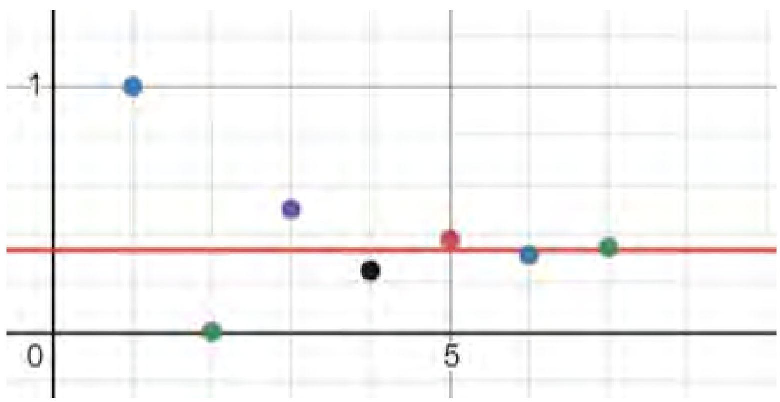

Figure 5: This graph shows the points from the table $(k, f(k))$ along with the line $y = \frac{1}{3}$, showing how the points seem to be converging to the line (created with desmos.com).

So, we now have a complete understanding of the relationship between the colors of vertices at fixed distances from each other in proper 3-colorings of the cycle as it becomes arbitrarily large. Now, we consider vertices at some distance away from each other that depends on m. This distance gets arbitrarily large as the cycle becomes arbitrarily large, as well.

Proposition 3.3. *Over all possible proper* 3*-colorings of* C_{2m} *and given some* $0 < x \leq 1$,

$$f(x) := \lim_{m\to\infty} P(v_1 \text{ and } v_{\lceil (m+1)x \rceil} \text{ are same color}) = \frac{1}{3}.$$

Proof. If $0 < x \leq 1$, then v_1 and $v_{\lceil (m+1)x \rceil}$ are more than distance two away as m goes to infinity, so we can split the larger cycle into two smaller cycles and compute

$$\begin{aligned} f(x) &= \lim_{m\to\infty} \frac{P_{C_{\lceil (m+1)x \rceil - 1}}(3) \cdot \frac{P_{C_{2m-\lceil (m+1)x \rceil+1}}(3)}{3}}{P_{C_{2m}}(3)} \\ &= \lim_{m\to\infty} \frac{\left(2^{\lceil (m+1)x \rceil - 1} + (-1)^{\lceil (m+1)x \rceil - 1} 2\right) \frac{2^{2m-\lceil (m+1)x \rceil+1} + (-1)^{2m-\lceil (m+1)x \rceil+1} 2}{3}}{2^{2m}+2} \end{aligned}$$

To be able to continue from here, suppose we just pick x so that $\lceil (m+1)x \rceil = (m+1)x$ since this does not change the answer as we take m to infinity, but does simplify the notation significantly. Then,

$$\begin{aligned} f(x) &= \lim_{m\to\infty} \frac{\left(2^{(m+1)x-1} + (-1)^{(m+1)x-1} 2\right) \frac{2^{2m-(m+1)x+1} + (-1)^{2m-(m+1)x+1} 2}{3}}{2^{2m}+2} \\ &= \lim_{m\to\infty} \frac{\left(2^{mx+x-1} + (-1)^{mx+x-1} 2\right)\left(2^{2m-mx-x+1} + (-1)^{2m-mx-x+1} 2\right)}{3\left(2^{2m}+2\right)} \\ &= \lim_{m\to\infty} \frac{2^{2m} + (-1)^{mx+x-1} 2^{2m-mx-x+2} + (-1)^{2m-mx-x+1} 2^{mx+x} + 4}{3 \cdot 2^{2m} + 6} \\ &= \lim_{m\to\infty} \frac{1 + (-1)^{mx+x-1} 2^{-mx-x+2} + (-1)^{2m-mx-x+1} 2^{mx+x-2m} + 2^{2-2m}}{3 + 3 \cdot 2^{1-2m}} \\ &= \frac{\lim_{m\to\infty}\left(1 + (-1)^{mx+x-1} 2^{-mx-x+2} + (-1)^{2m-mx-x+1} 2^{mx+x-2m} + 2^{2-2m}\right)}{\lim_{m\to\infty}\left(3 + 3 \cdot 2^{1-2m}\right)} \\ &= \frac{1+0+0+0}{3} = \frac{1}{3} \end{aligned}$$

because $\lim_{m\to\infty} 2^{-mx-x+2} = 0$ and $\lim_{m\to\infty} 2^{mx+x-2m} = 0$ since $0 < x \leq 1$, so $2m > xm$. Therefore, $f(x) = \frac{1}{3}$ for all $0 < x \leq 1$.

□

Thus, the colors of v_1 and $v_{\lceil (m+1)x \rceil}$ are asymptotically independent for all $0 < x \leq 1$. For example, this means that, as C_{2m} gets arbitrarily large, a vertex v and its diametrically opposing vertex u have $\frac{1}{3}$ chance of being the same color, and the same is true for any vertex at distance ϵm away from v, for any $\epsilon > 0$. The color of v has no effect on the color of these farther away vertices.

Additionally, we conjecture the following.

Conjecture 3.4. *For C_{2m}, we claim that for any $\epsilon \in \mathbb{R}$,*

$$P\left(\left|\frac{\#\text{ red vertices}}{2m} - \frac{1}{3}\right| > \epsilon\right) \to 0$$

as $m \to \infty$.

While Conjecture 3.4 is still in the process of being proven using either Chernoff inequalities or some local central limit theorem, we do have a general understanding of

$$\lim_{m\to\infty} P\,(\text{one third of vertices are red})$$

for C_{2m}. While this limit is equal to zero, the process through which we arrive at this conclusion provides some insights. First, however, we will have to understand how many *independent sets*, sets of vertices where no pair of vertices share an edge, of specific sizes there are in a given cycle.

Remark 3.5. Given a cycle C_a with a vertices, we want to know how many independent sets there are of size b for some $0 < b < a$. If we label one vertex of the cycle v, we first compute how many b-vertex independent sets there are in C_a that contain v. If we include v, then we cannot include either of its neighbors, and are left with a path of $a-3$ vertices from which we want to choose $b-1$ independent vertices. Using a common combinatorial strategy, this is equivalent to picking $b-1$ vertices from a path of $(a-3)-(b-2)$ vertices, and then insert $b-2$ spacer vertices between each vertex we picked to ensure that the set is independent. There are $\binom{(a-3)-(b-2)}{b-1} = \binom{a-b-1}{b-1}$ ways to do this. Now, we will count the number of pairs (i, I) where I is an independent set of b vertices in C_a and $i \in V(I)$. If pick the vertex i first, then the independent set I, we have $a\binom{a-b-1}{b-1}$ ways to do this; if we pick I first, and then i, there are $|\{I \mid I \text{ is a } b\text{-vertex independent set in } C_a\}|b$ ways to do this. This gives us

$$a\binom{a-b-1}{b-1} = \big|\{I : I \text{ is a } b\text{-vertex independent set in } C_a\}\big| \cdot b,$$

and thus that

$$\big|\{I : I \text{ is a } b\text{-vertex independent set in } C_a\}\big| = \frac{a}{b}\binom{a-b-1}{b-1}.$$

Thus, there are $\frac{a}{b}\binom{a-b-1}{b-1}$ b-vertex independent sets in C_a.

For ease of notation, let $m = 3k$ for some $k \in \mathbb{N}$ (so we are just considering cycles with a multiple of 6 many vertices). To count the number of colorings where $2k$ vertices (one third of the vertices) are red, we start by choosing an independent set in the cycle with exactly $2k$ vertices that we will color red. As seen in Remark 3.5, there are $\frac{6k}{2k}\binom{6k-2k-1}{2k-1} = 3\binom{4k-1}{2k-1}$ ways to do this. Then, to color the rest of the vertices, we just have to pick one of two colors for each vertex to the right of a red vertex, and the rest of the colors will be determined. There are $2k$ red vertices, so 2^{2k} ways to do this.

This gives us $3 \cdot 2^{2k}\binom{4k-1}{2k-1}$ total ways to color a $6k$-cycle with a third of its vertices colored

red, while we know that the total number of 3-colorings of this cycle is $2^{6k}+2$. Thus,

$$\begin{aligned}
\lim_{m\to\infty} P\left(\frac{1}{3} \text{ vertices red}\right) &= \lim_{k\to\infty} \frac{3\cdot 2^{2k}\binom{4k-1}{2k-1}}{2^{6k}+2} && \text{(see above)} \\
&\le \lim_{k\to\infty} \frac{3\cdot 2^{2k}\binom{4k}{2k}}{2^{6k}+2} && \text{(Pascal's)} \\
&\sim \lim_{k\to\infty} \frac{3\cdot 2^{2k}\frac{2^{4k}}{\sqrt{2k\pi}}}{2^{6k}+2} && \text{(Stirling's approximation)} \\
&= \lim_{k\to\infty} \frac{2^{6k}}{2^{6k}+2}\cdot\frac{3}{\sqrt{2\pi k}} \\
&= \lim_{m\to\infty} \frac{2^{2m}}{2^{2m}+2}\cdot\frac{3}{\sqrt{\frac{2}{3}\pi m}} = 1\cdot 0 = 0,
\end{aligned}$$

so $\lim_{m\to\infty} P\left(\frac{1}{3} \text{ vertices red}\right) = 0$ since a probability cannot be negative. We should expect this probability to have limit zero since the event consists of exactly $\frac{1}{3}$ of the vertices being red, as opposed to the proportion of red vertices just being close to $\frac{1}{3}$ (as in Conjecture 3.4). What is notable about this limit, then, is not that it equals zero, but that it is not the exponential terms that cause it to go to zero (as 2^{2m} and $2^{2m}+2$ effectively cancel each other out), it is the term $\frac{3}{\sqrt{\frac{2}{3}\pi m}}$ of order $\frac{1}{\sqrt{m}}$. This enumeration can be generalized to any number j of red vertices, not just $2k$, with similar results. This order $\frac{1}{\sqrt{m}}$ decay suggests that the standard deviation of the number of red vertices is order $\sqrt{m}$. If this is true, then the order $\sqrt{m}$ standard deviation of the number of red vertices grows much more slowly than the $2m$ total vertices in C_{2m}, and thus will imply a result similar to our Conjecture 3.4.

Conjectures aside, we know from Proposition 3.3 that the colors of vertices at fixed fractions of C_{2m} away from each other are indeed independent as m goes to infinity. As we will see in the next section, this is in stark contrast to the behavior of proper 3-colorings on the hypercube, where the colors of different pairs of vertices are never asymptotically independent.

4 3-colorings of hypercubes

Now, we investigate similar questions for the hypercube Q_n. However, exact enumeration, as used for cycles in the last section, is not a viable approach for hypercubes, since it is just too hard to directly count the total number of proper 3-colorings of the hypercube with specific properties. Instead, we consider a new probability distribution $\hat{\mu}$ from [5] that approximates the uniform proper coloring distribution μ that we used in Section 3 for cycles. Understanding $\hat{\mu}$ requires background on graph homomorphisms, dominant patterns in the coloring, and sets of vertices that deviate from that pattern called defect vertices. While the background and definition for $\hat{\mu}$ are very complicated, the individual steps that comprise it can be understood probabilistically. In fact, [5] are able to show that $\hat{\mu}$ is actually very close to μ for sufficiently large n, so close that they are able to use $\hat{\mu}$ to prove a result for μ that then allows us to compute the probabilities that we are interested in.

After building up these tools, we use them to prove that, indeed, the structure and bipartite nature of the hypercube do result in a uniformly randomly chosen proper 3-coloring of the hypercube generally just looking like a 2-coloring. Informally, this means that almost all vertices in $\mathcal{E}$ are one color, while almost all vertices in $\mathcal{O}$ are one of the two other colors, or vice versa.

4.1 Background from Jenssen and Keevash

The family of graphs considered in [5] is $\mathbb{Z}_m^n$, the n-dimensional discrete torus. Q_n, the hypercube of dimension n, is just $\mathbb{Z}_2^n$, so we can easily interpret the results in [5] as results for Q_n by setting $m = 2$.

Additionally, [5] focus on weighted homomorphisms from the discrete torus to an arbitrary fixed graph H. While we will ignore the weighted aspect of the homomorphisms in this paper entirely and set $H := K_3$, we will want to explain how $\mathrm{Hom}(Q_n, K_3)$, the set of graph homomorphisms from Q_n to K_3, is also just the set of proper 3-colorings of Q_n. (Note that the *complete graph*, K_n, consists of n vertices and all possible edges between them, so K_3 just looks like a triangle and models 3-coloring.) A *graph homomorphism* from one graph to another is a function that maps vertices from the first graph to the second in a way that preserves adjacency structure (vertices that were neighbors in the first graph are still neighbors in the second), as seen in Figure 6. Thus, a graph homomorphism from any graph G to K_3 shows us exactly how to color the vertices in G with three colors so that no adjacent vertices have the same color.

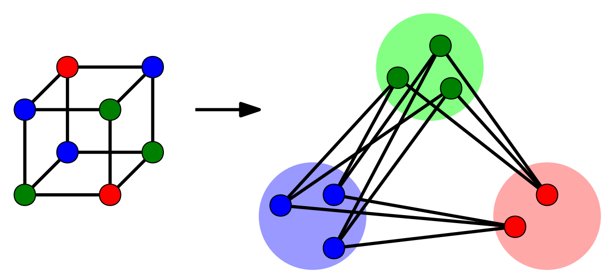

Figure 6: An example of a graph homomorphism from Q_3 to K_3 is shown.

Going forward, results cited from [5] will be specialized to apply specifically to Q_n and proper 3-colorings.

The uniform probability measure on the set of proper 3-colorings of some hypercube Q_n is

$$\mu(f) := \frac{1}{|\mathrm{Hom}\,(Q_n, K_3)|}.$$

However, it is difficult to find the size of $|\mathrm{Hom}\,(Q_n, K_3)|$ (the number of total proper 3-colorings of G) directly for hypercubes, so we will need to build up some more tools to approach this problem from a different direction.

First, we note that Q_n is bipartite, so we denote its vertex classes by $\mathcal{E}$ and $\mathcal{O}$. Given $A, B \subseteq V(K_3)$, the pair (A, B) is a *pattern* if $\{a, b\} \in E(K_3)$ for all $a \in A$ and $b \in B$. Since this is always true in K_3 as long as $a \neq b$, a pair (A, B) is a pattern if and only if A and B are *disjoint* subsets of vertices in K_3. We call a pattern (A, B) *dominant* (for proper 3-colorings of hypercubes specifically) if $|A| \cdot |B| = 2$. Thus, if we let $V(K_3) = \{v_1, v_2, v_3\}$, then there are $\binom{3}{1} + \binom{3}{2} = 6$ total dominant patterns:

$$\begin{aligned} &(\{v_1\}, \{v_2, v_3\}), && (\{v_1, v_2\}, \{v_3\}), \\ &(\{v_2\}, \{v_1, v_3\}), && (\{v_1, v_3\}, \{v_2\}), \\ &(\{v_3\}, \{v_1, v_2\}), \text{ and} && (\{v_2, v_3\}, \{v_1\}). \end{aligned}$$

Let $\mathcal{D}(K_3)$ denote the collection of all dominant patterns shown above.

We call a proper 3-coloring $f \in \mathrm{Hom}(Q_n, K_3)$ a *dominant coloring* if $f(\mathcal{E}) \subseteq A$ and $f(\mathcal{O}) \subseteq B$ for some dominant pattern (A, B). In a dominant proper 3-coloring of Q_n, this means that identifying a specific two of the three total colors would turn the proper 3-coloring into a proper 2-coloring.

What we will show later is that a uniformly randomly chosen proper 3-coloring of the hypercube will generally be very close to being a dominant coloring – most vertices will follow the dominant pattern except for a small number of *defect vertices.*

4.2 Defect vertices

With that in mind, we need to find a way to understand how many defect vertices we might expect to find. One definition that we will need from [2] in arriving at this understanding is the *square* of a graph G, denoted G^2. G^2 is a graph with the same vertex and edge sets as G, but with additional edges added between each pair of vertices at distance two from each other in G. Then, a subset of vertices $\gamma \subseteq V(G)$ is *G^2-connected* if the corresponding *induced subgraph* of G^2 is connected. (The induced subgraph has vertex set γ and all edges from $E(G^2)$ that involve only vertices in γ.)

While [5] goes much more into depth on defect vertices, all we will need to know is that a *polymer* $\gamma \subseteq V(G)$ is $(Q_n)^2$-connected and relatively small, a *polymer configuration* Γ represents a set of *mutually compatible* polymers (a set of polymers such that the distance between any polymer in Γ is greater than two), and Ω is the set of all mutually compatible polymers. These sets of polymers represent the sets of defect vertices we will find, a.k.a. the few vertices that will disagree with the dominant pattern. Then, $\nu_{A,B}$ is a probability distribution on Ω that is mentioned in Definition 4.1, but the details of which will not concern us.

Now, to understand how this fits into the probability of a random vertex in a proper 3-coloring of Q_n being a defect vertex or not, we consider the following probability measure $\hat{\mu}$. While this probability measure is not the standard probability measure μ, it does approximate μ, as we will see later.

Definition 4.1 (Jenssen and Keevash [5])**.** For $f \in \text{Hom}(Q_n, K_3)$, let $\hat{\mu}(f)$ denote the probability that f is selected by the following four-step process:

1. Choose $(A, B) \in \mathcal{D}(K_3)$ with uniform probability. (Recall that there are six possibilities, so $\frac{1}{6}$ probability each.)

2. Choose a random polymer configuration $\Gamma \in \Omega$ (the set of mutually compatible sets of polymers) from the distribution $\nu_{A,B}$.

3. Letting $S = \bigcup_{\gamma \in \Gamma} \gamma$, select a coloring $f \in \hat{\chi}_{A,B}(S)$ with uniform probability. ($\hat{\chi}_{A,B}(S)$ is the set of colorings of just S and its neighbors that always disagree with (A, B) on S and agree with (A, B) on $S^+ \setminus S$ where S^+ denotes the set of vertices in S as well as their neighbors.)

4. Independently assign each $v \in \mathcal{O} \setminus S^+$ the color $i \in A$ with probability $\frac{1}{|A|}$ and each $v \in \mathcal{E} \setminus S^+$ the color $j \in B$ with probability $\frac{1}{|B|}$. (We know that either $|A| = 1$ and $|B| = 2$ or vice versa.)

In this definition, S is the set of defect vertices, and we are able to color S and its neighbors separately from the rest of the vertices. This is true since, if a vertex and all of its neighbors v follow the dominant pattern, the set of color options for v is entirely disjoint from the set of color options for its neighbors, so we can assign a color to v independent of the colors of its neighbors. This means that the only vertices that will be complicated to color will be the defect vertices and their neighbors. The following remark gives us a bound on how many vertices that will be.

Remark 4.2. Each vertex in Q_n has a maximum of n neighbors, so this gives us the following naïve, but useful, bound:

$$|S^+| \leq |S| \cdot (n + 1).$$

As for the probability distribution $\hat{\mu}$ more broadly, [5] proved that, indeed, $\hat{\mu}$ does approximate μ as n goes to infinity.

Theorem 4.3 (Jenssen and Keevash [5])**.** *The measures μ and $\hat{\mu}$ on* $\text{Hom}(Q^n, K_3)$ *satisfy*

$$\|\hat{\mu} - \mu\|_{TV} \leq e^{-\Omega\left(\frac{2^n}{n^2}\right)}.$$

This means that the *total variation distance* of μ and $\hat{\mu}$, which is $\sup_A |\hat{\mu}(A) - \mu(A)|$, the supremum of the difference between them for all possible events A, is bounded above by $e^{-\Omega\left(\frac{2^n}{n^2}\right)}$. As defined in [4],

$$f(n) = \Omega(g(n)) \iff |f(n)| \geq C|g(n)|$$

for some $C > 0$, so Ω gives a type of lower bound, so the difference between the probabilities of μ and $\hat{\mu}$ for any event is either $e^{-C \cdot \frac{2^n}{n^2}}$ or less. As n goes to infinity, 2^n grows much faster than n^2 and C is just a constant, so the difference between the probability distributions on any event goes to zero.

However, we will not actually have to make use of $\hat{\mu}$ since [5] used Theorem 4.3 to prove a result about the number of defect vertices we should expect to find in a hypercube, specifically. With this understanding of the defect vertices and structure of a dominant coloring, computing probabilities becomes much simpler.

Corollary 4.4 (Jenssen and Keevash [5]). *Let X_n denote the number of defect vertices in a uniformly chosen proper 3-coloring of Q_n. Then, $X_n \xrightarrow{d} Pois(1)$ where '$\xrightarrow{d}$' denotes convergence in distribution as $n \to \infty$.*

$Pois(1)$, the *Poisson distribution* with parameter $\lambda = 1$, is a famous and well understood discrete probability distribution. The *probability mass function* of $Pois(1)$, which is the function $P(Y = k)$ that tells us the probability that a random variable Y with distribution $Pois(1)$ will take on the value k, is

$$P(Y = k) = \frac{e^{-1}}{k!}.$$

Crucially, both the *expected value* and *variance* of Y are one. In fact, there is a probability of $\frac{1}{e}$ that Y will be zero and the same for one, so there is actually a probability of $\frac{2}{e} \approx .74$ that $Y < 2$. Similarly, if each vertex independently had a probability $\frac{1}{2^n}$ of being a defect vertex (recall that 2^n is the number of vertices in Q_n), then the number of defect vertices would also converge in distribution to $Pois(1)$ as $n \to \infty$. All this to say, this corollary implies that the number of defect vertices is very small.

Since we know that $X_n \xrightarrow{d} Pois(1)$, Theorem 10.0.1 in [6] tells us that, supposing Y is a random variable with distribution $Pois(1)$,

$$P(X_n \in A) \to P(Y \in A) \text{ as } n \to \infty$$

for all measurable sets (or *events*) $A \subseteq \mathbb{R}$ such that $P(Y \in \partial A) = 0$ where ∂A is the *boundary* of A. Since the only kinds of events that we will be considering are intervals, the boundary of an event A will just consist of the two values on either end of the interval (excluding $-\infty$ and ∞).

Corollary 4.4 and this theorem imply that the number of defect vertices is essentially constant as the hypercubes grow in size, which is remarkable. We need one final result to formalize this idea.

Proposition 4.5. *If X_n is the number of defect vertices in a uniformly chosen proper 3-coloring of Q_n, then $\lim_{n\to\infty} P(X_n \leq n) = 1$.*

Proof. Since $\lim_{n\to\infty} P(X_n \leq n) = 1$ is equivalent to $\lim_{n\to\infty} P(X_n > n) = 0$, we prove the latter instead. Then, for any $\epsilon > 0$, we will find some $N_\epsilon > 0$ such that $n > N_\epsilon$ implies that $P(X_n > n) < \epsilon$.

To do so, we start by taking M_0 such that $P(Pois(1) > M_0) < \frac{\epsilon}{2}$. Note that we can do this since we already know that $\lim_{n\to\infty} P(Pois(1) > n) = 0$, so M_0 is a large integer that depends on ϵ. Then, by Theorem 10.0.1 from [6] (see above), we know that

$$\lim_{n\to\infty} P(X_n > M_0) = P(Pois(1) > M_0) < \frac{\epsilon}{2}.$$

(Note that both X_n and $Pois(1)$ are discrete so we can just add $\frac{1}{2}$ to M_0 and the probabilities don't change, but the probability of the boundary becomes zero.) Now, we choose M_1 such that $P(X_n > M_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > M_1$ and $M_1 > M_0$ if it isn't already. We can find an M_1 such that $P(X_n > M_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > M_1$ since there exists some M_1 such that $\left|P(X_n > M_0) - \frac{\epsilon}{2}\right| < \frac{\epsilon}{2}$ for all $n > M_1$ since $\lim_{n\to\infty} P(X_n > M_0) < \frac{\epsilon}{2}$. Additionally, if $M_1 < M_0$, we can just set $M_1 := M_0 + 1$ instead since making M_1 larger does not change the previous inequalities.

Now, we have that $n > M_1 > M_0$, so

$$P(X_n > n) < P(X_n > M_0) < \epsilon$$

because $n > M_1$. Thus, $N_\epsilon = M_1$ and $\lim_{n\to\infty} P(X_n > n) = 0$, so we're done!

□

This gives us a bound on the number of defect vertices in Q_n as $n \to \infty$ that will be very useful in the next section.

4.3 Probabilities

Now that we've set up the necessary machinery, we can answer some of our questions about the probability of pairs of vertices at various distances being the same color. To start, we find the probability that two vertices at some fixed distance x from each other the same color as the hypercube gets arbitrarily large.

Proposition 4.6. *Over all possible proper 3-colorings of Q_n and given some fixed integer $x > 0$,*

$$f(x) := \lim_{n\to\infty} P(2 \textit{ vertices at distance } x \textit{ are same color}) = \begin{cases} \frac{3}{4}, & x \textit{ is even,} \\ 0, & x \textit{ is odd.} \end{cases}$$

Proof. To find this probability, we compute

$$\begin{aligned} &P(u, v \in V \setminus S^+) \cdot P(\text{same color} \mid u, v \in V \setminus S^+) \\ &\quad + P(u \in S^+ \text{ and/or } v \in S^+) \cdot P(\text{same color} \mid u \in S^+ \text{ and/or } v \in S^+), \end{aligned}$$

as $n \to \infty$ and for uniformly randomly chosen u, v at distance x from each other. Now, we consider these probabilities separately:

- $\underline{P(u, v \in V \setminus S^+)}$: First, we count the number of ways to uniformly randomly choose a pair of points (u, v) at distance x from each other that involve at least one vertex in S^+. Given a u, there are $\binom{n}{x}$ options for v, so a set of defect/neighbor vertices of size $|S^+| = k$ can have a maximum of $k\binom{n}{x}$ total pairs of this type that have at least one vertex in S^+. There are $\frac{2^n\binom{n}{x}}{2}$ total pairs of this type in Q_n. We know that $P(k \leq n(n+1)) = 1$ as $n \to \infty$ by Proposition 4.5 and Remark 4.2, so the number of pairs of vertices involving S^+ must be less than or equal to $n(n+1)\binom{n}{x}$. Then, since

$$\lim_{n\to\infty} \frac{n(n+1)\binom{n}{x}}{\frac{2^n\binom{n}{x}}{2}} = 0$$

(found using repetitions of L'Hôpital's rule as $\binom{n}{x}$ is polynomial in n), it must be true that

$$\lim_{n\to\infty} \frac{|\{\text{all pairs of vertices involving } S^+\}|}{|\{\text{all pairs at distance } x\}|} = 0,$$

as well by the Squeeze Theorem. Therefore, as $n \to \infty$, the probability that a uniformly randomly selected pair of vertices (u, v) at distance x has neither vertex be a defect vertex (or neighbor of a defect vertex) is one. Therefore,

$$P(u, v \in V \setminus S^+) = 1.$$

- $\underline{P(\text{same color} \mid u, v \in V \setminus S^+)}$: Since $u, v \in V \setminus S^+$, they *and* their neighbors must agree with the dominant pattern. If the distance x between u and v is even, then we have a $\frac{1}{2}$ probability that the dominant pattern gives one option for u, v and a $\frac{1}{2}$ probability that it gives two options (since all the neighbors are in the dominant pattern too, both colors are possible). This gives us that

$$\begin{aligned} P(\text{same color} \mid u, v \in V \setminus S^+) &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{3}{4}. \end{aligned}$$

If the distance x between u and v is odd, the dominant pattern guarantees that they will be different colors, so

$$P(\text{same color} \mid u, v \in V \setminus S^+) = 0.$$

- $\underline{P(u \in S^+ \text{ and/or } v \in S^+)}$: This value is just $1 - P(u, v \in V \setminus S^+)$, so, zero as $n \to \infty$.

- $\underline{P(\text{same color} \mid u \in S^+ \text{ and/or } v \in S^+)}$: Since $P(u \in S^+ \text{ and/or } v \in S^+) = 0$ as $n \to \infty$, it doesn't matter what this value is since we know that it is bounded above by 1 and below by zero.

Therefore, we have that

$$\lim_{n\to\infty} P(2 \text{ vertices at distance } x \text{ are same color}) = 1 \cdot \frac{3}{4} + 0 \cdot ? = \frac{3}{4}$$

when x is even and 0 otherwise.

$\square$

This means that, regardless of how far apart two vertices are in an arbitrarily large hypercube, if the distance between them is even, they are likely the same color (depending on which half of the dominant pattern they are in), and if the distance between them is odd, they are almost definitely not the same color. This result is in stark contrast to our results for the cycle, where the color of one vertex was completely independent of the color of another vertex a fixed fraction of the cycle far away. Additionally, while vertices at a fixed distance from each other had varying probabilities of being the same color in the cycle (with this probability going to $\frac{1}{3}$ as the distance between them became arbitrarily large), the probability of two vertices in the hypercube being the same color depends solely on the parity of the distance between them, regardless of how small or large it is.

Finally, we want to get a better idea of what these colorings look like, beyond just the relationship between individual pairs of vertices. The following conjecture is still being worked out using strategies from [5], but seems very likely to be true.

Conjecture 4.7. *For Q_n, we claim that for any sufficiently small, positive $\epsilon \in \mathbb{R}$ (for example, any $\epsilon < \frac{1}{24}$ so that the following intervals are disjoint),*

$$P\left(\left|\frac{\#\text{ red vertices}}{2^n} - \frac{1}{3}\right| < \epsilon\right) \to 0,$$

$$P\left(\left|\frac{\#\text{ red vertices}}{2^n} - \frac{1}{2}\right| < \epsilon\right) \to \frac{1}{3}, \text{ and}$$

$$P\left(\left|\frac{\#\text{ red vertices}}{2^n} - \frac{1}{4}\right| < \epsilon\right) \to \frac{2}{3}$$

as $n \to \infty$.

Crucially, this means that the probability that about $\frac{1}{3}$ of the vertices are one color is zero in an arbitrarily large hypercube, whereas we conjectured in Conjecture 3.4 that the probability that about $\frac{1}{3}$ of the vertices are one color is one for an arbitrarily large cycle. Instead, for hypercubes, we will have about $\frac{1}{2}$ of the vertices be red $\frac{1}{3}$ of the time, and about $\frac{1}{4}$ will be red the rest of the time.

5 Conclusion

In this paper, we have shown the difference in proper 3-colorings of arbitrarily large cycles and hypercubes. While we were able to fully understand colorings of even cycles by direct enumeration and found that the colors of vertices a fixed fraction of the cycle away from each other are independent of each other as the cycle becomes arbitrarily large, the same is not true for hypercubes. Since direct enumeration was not viable for understanding proper 3-colorings of arbitrarily large hypercubes, we instead adapted results from [5] to bound the number of defect vertices (vertices that didn't fit into the dominant pattern). Since the dominant pattern always colored the even vertices one color and the odd vertices a mix of the other two colors (or vice versa), this allowed us to prove definitively that the color of vertices at arbitrary distances from each other in the arbitrarily large hypercube are not independent. If they are the same parity, then depending on the part of the dominant coloring they are in, they either have a $\frac{1}{2}$ probability of being the same color or a probability of 1 of being the same color; if they are not the same parity, then there is 0 probability that they are the same color. In this way, there is a clear structure to a uniformly randomly chosen proper 3-coloring of Q_n and $n \to \infty$, where the same is not true for C_n.

Graphs model networks, and so these two specific cases model how local constraints can propagate very differently through different kinds of networks. Additionally, this helps to show that the results from [5] are not just interesting theoretically, but can be applied in practice to approximate specific probabilities for hypercubes (the smallest non-trivial discrete torus).

As for future work, the proof of both Conjectures 3.4 and 4.7 would help significantly in formalizing the intuition of what the distribution of colors in these proper 3-colorings should look like. Approximating similar probabilities for larger dimensional discrete tori would be of interest, as well.

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