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Payment Schemes and Moral Hazard

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1 Introduction

In a principal-agent relationship, the principal offers a take-it-or-leave-it contract to the agent, who decides to either accept it or not. In game theory terminology, the principal-agent relationship is a Stackelberg game in which the principal is the leader, proposing the contract, and the agent is the follower, choosing to accept or reject the proposal. Examples of such relationships are plentiful, such as a principal bank manager hiring an agent employee to work as a teller, a principal land-owner acting hiring an agent farmer to grow crops on her land, or an insurance company offering a home insurance plan to a homeowner. The principal-agent problem concerns how the principal should structure the proposed contract to best incentivize the agent to perform in the way the principal would prefer, taking into account that there are informational asymmetries between the principal and the agent due to the agent having some kind of "private information." Information asymmetries between principal and agent fall into two categories: the agent might have private information about their own characteristics, which gives rise to adverse selection problems; or the agent might have private information about what actions he takes after agreeing to the contract, which gives rise to moral hazard problems. In this paper, I focus on a model with moral hazard. The texts by Kreps [3] and Salanié [5] both offer good expositions of canonical adverse selection and moral hazard problems, which I used as a starting point for this paper. The survey of different extensions of the principal-agent model by Sappington [6] provided a high-level guide to different sub-problems and primary sources.

To motivate the model analyzed in this paper, suppose you own several tracts of land that are suitable for agriculture. You want to set up farms on these tracts of land, but you lack the time or expertise to farm the land yourself. You decide, then, to hire several farmers to set up and manage farms on your land. The farmers work year-round and, come harvest time, you pay each of them a sum of money based on their total production. Your challenge is to decide how much money to pay each farmer. Ideally, you would like to be able to pay each worker for the amount of effort that they put in. Unfortunately, you are only able to observe each farmer's output, and there are factors other than the farmer's effort level that affect output. For instance, the amount of rainfall is a random variable that affects all farmers' output equally, but which you are unable to observe. There are also idiosyncratic random variables unique to each farmer that represent the effects of soil condition, pests, and other similar concerns on the tract of land that farmer is working. All else equal, each farmer would prefer to work as little as possible, because they find working displeasurable. As the principal, however, you want the farmers to work as hard as is necessary to maximize your profits. The problem you face is how to structure the farmers' payment scheme so as to align their incentives with your own.

The following analysis will compare individual contracts, in which each agent's payment is based only on the realized magnitude of their output, and tournament payment schemes, in which each agent's payment is based only on the ordinal ranking of their realized output relative to that of all other agents'. Much of the model notation as well as the results from Section 5 are an expanded exposition of results from a paper by Green and Stokey [1]. Lazear and Rosen [4] provided helpful intuition for the comparison of contracts and tournaments, and some of the results from earlier sections of the paper are due to Grossman and Hart [2].

2 Model Definitions

The principal-agent problem, in general terms, concerns how to align the incentives of a rational, self-interested agent party with the interests of a different principal party. In the problem described above, the land-owner is the principal and each of the hired farmers are agents. In this section I define in precise terms the nature of this principal-agent problem.

Each agent i chooses some action level $a_i \in \mathbb{R}_+$ which can be interpreted as an effort level. In general, the principal wants the agent to exert more effort by choosing a higher action level, which increases production, while the agent would like to choose a lower action level. The principal makes a payment of $m_i \in \mathbb{R}_+$ to agent i.

For each agent i, there is a utility function

$$U^i(m_i, a_i) = u(m_i) - a_i,$$

where $u : \mathbb{R}_+ \to [0, B]$, $u(0) = 0$, is a function that defines how much utility the agent gains from the payment he receives. It is natural to assume that this function is strictly increasing, $\mathrm{d}u > 0$, since if an agent receives more money he will be afford more things to satisfy his needs and desires. We will also assume that the function is strictly concave, $\mathrm{d}^2u < 0$. This assumption, called the law of diminishing marginal utility, is common in economic analysis. The intuition is essentially that the less money a person has, the more valuable a given increase in their wealth is to them. A person living in poverty would benefit much more from a million-dollar windfall than a person who's net worth is already in the billions of dollars.

This formulation assumes that utility is additively separable into a utility of payment term and a disutility of effort term. This means that the marginal utility of an increase in payment is constant in action level, and vice versa; it does not allow for how hard an agent is working to affect how that agent values a payment, nor does it allow how much an agent is being paid to have an affect on the disutility of effort for that agent.

Each agent additionally has a reservation utility $u^0 > 0$ representing the agent's expected utility of not entering into a contract with the principal. This is the expected utility of whatever the agent's next-best use of their time and effort is.

For each agent i there is a production function that gives y_i, the gross revenue generated by agent i

$$y_i = z_i + \theta,$$

where $\theta \in \mathbb{R}$ is a random variable affecting the production of every agent–a "common shock"–with $E[\theta] = 0$, and z_i is a random variable whose distribution depends on a_i. θ represents the state of nature that affects the production levels of all agents, such as the amount of rainfall or the average temperature in a given year. The z_i contains the part of agent i's production that is due to idiosyncratic factors, including agent i's choice of action. Let $F(\ ; a_i)$ be the cumulative distribution function of z_i given that action a_i was taken, and let $f(\ ; a_i)$ be the continuously differentiable density function. So if agent i takes action a_i, the probability that the idiosyncratic part of their production is equal to or less than some level z_i is $F(z_i; a_i)$, and the likelihood that it is very near z_i is $f(z_i; a_i)$.

Each agent observes a signal $\sigma_i \in \mathbb{R}$ about the value of θ. Let $G(\theta, \sigma)$ be the joint distribution function of θ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. G summarizes the agents' information about the distribution of θ. We assume that z_i and (θ, σ) are independent. This formulation allows for cases where each agent receives the same signal and is perfectly informed of the value of θ by the signal they receive, i.e. when $\theta = \sigma_1 = \sigma_2 = \ldots = \sigma_n$ with probability 1. It also allows for cases where the signals agents observe aren't necessarily the same and contain no useful information about the value of θ, when

$$G(\theta|\sigma) = G(\theta|\sigma') \quad \forall \sigma, \sigma' \in \mathbb{R}^n,$$

and for many situations between in which agents are partially but imperfectly informed about the value of θ. The one restriction that this specification of $G(\theta, \sigma)$ does place on our model is that it assumes all agents' information sets are identical except for the signal they receive about θ.

We assume that all agents and the principal know the distributions G, F, and f. Each agent i is aware of their choice of action a_i and of the signal σ_i. The agents do not know one another's choice of action or signal, and the principal does not know any agent's choice of action or signal.

2.1 Agent's Problem

Agent i is faced with the following problem: what level of action should they take in order to maximize their total expected utility. In the most general formulation, we can characterize this problem as follows. Agent i is faced with some reward function $R_i : \mathbb{R}^n \to \mathbb{R}_+$ that maps vectors of the productions of all n agents, $\mathbf{y} \equiv (y_1, y_2, \ldots, y_n)$, to m_i, the amount of compensation agent i receives when the production levels $y_1, \ldots, y_n$ are realized.

Definition 2.1. Define $v_i : \mathbb{R}^n \to [0, B]$ as $v_i(\mathbf{y}) \equiv u[R_i(\mathbf{y})]$, the payment to agent i in terms of utility. Then, the cost to the principal of providing this level of utility is $\gamma[v_i(\mathbf{y})]$, where $\gamma \equiv u^{-1}$. Since u is strictly increasing, it must be a bijection, and so we know that the inverse, γ, exists.

Then, the agent must choose an action $a_i(R_i)$ that is a solution to the maximization problem

$$\operatorname*{argmax}_a E[v_i(\mathbf{y}) - a] = \operatorname*{argmax}_a \int_{\mathbf{y} \in \mathbb{R}^n} v_i(\mathbf{y}) P(\mathbf{y}|a) \, \mathrm{d}\mathbf{y} - a, \tag{2.1}$$

where $P(\mathbf{y}|a)$ is the likelihood that the production vector $\mathbf{y}$ occurs, given that agent i has chosen action a. In words, the agent seeks to choose an action that maximizes the expected utility of the payment he will receive net the disutility of his chosen action level.

2.2 Principal's Problem

We assume that the principal is risk neutral, or similarly that if the principal is risk averse he is able to diversify away risk by engaging in similar relationships with multiple agents. The principal, then, wants to maximize total profits–the expected value of the sum of all agent's outputs minus the total of all payments made to them–given that the agent is going to take action $a_i(R)$ that solves problem (2.1). The principal seeks to design a payment scheme $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ to maximize

$$E\left[\sum_{i=1}^{n} [y_i - R_i(\mathbf{y})] \,\middle|\, a_i(R_i) \right]. \tag{2.2}$$

In this paper, we will examine two different kinds of payment schemes that the principal can choose to use. One payment scheme is individual contracts. Under an individual contract, each agent's payment depends only on his own level of output. An example of such a payment scheme would be a piece-rate wage, in which a worker is paid a fixed amount for each unit of production. Individual contracts can also be more complicated than a simple piece-rate, but always depend on only the level of output achieved by the agent. Under a tournament payment scheme, each agent's payment depends only on the ordinal ranking of his output relative to that of the other agents. The agent who has the highest output receives a certain payment, regardless of what magnitude of output that is.

3 First Best

In the first best solution to this principal-agent problem, the principal can perfectly observe the agent's choice of action a. Moral hazard is not an issue in the first best case, because the agent's chosen action is not hidden information. The principal can offer contracts in which payment depends not on observed output, but instead on a.

Definition 3.1. Let $C_{FB} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as $C_{FB}(a) \equiv \gamma[a + u^0]$. $C_{FB}(a)$ is an agent's reservation price for taking action a in the first-best situation.

For any action a^*, the principal can offer an agent a contract in which the agent's compensation, m, is

$$m = \begin{cases} C_{FB}(a^*) & \text{if } a = a^* \\ 0 & \text{if } a \neq a^*. \end{cases}$$

Under this contract, if the agent takes some action other than a^* his net utility is

$$\begin{aligned} U^i(m, a) &= u(0) - a \\ &= -a && \text{(since } u(0) = 0) \\ &\leq 0, \end{aligned}$$

but if he takes action a^*, his net utility is

$$\begin{aligned} U^i(m, a^*) &= u[\gamma(a^* + u^0)] - a^* \\ &= a^* + u^0 - a^* \\ &= u^0 \\ &> 0. \end{aligned}$$

Because $U^i(m, a^*) > U^i(m, a)\ \forall a \neq a^*$, the agent will always choose $a^*$to solve problem (2.1). So in the first best case, the principal can force the agent to take any action a at an expected cost of $C_{FB}(a)$.

Definition 3.2. Let $B(a) : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as $B(a) \equiv \int y f(y; a)\, \mathrm{d}y$. $B(a)$ is the expected revenue for an agent taking action a.

Then, in the first best situation, the principal's problem reduces to maximizing profit– the difference between $B(a)$, the expected revenue from an agent taking action a, and $C_{FB}(a)$, the cost of getting an agent to take action a:

$$\begin{aligned} P_{FB} &\equiv \max_a [B(a) - C_{FB}(a)] \\ &= \max_a \int y f(y; a)\, \mathrm{d}y - \gamma(a + u^0) \end{aligned}$$

Above, P_{FB} is the principal's expected profits in the first best case. Call the argument maximizing the above problem a_{fb}, the first best efficient action level.

4 Second Best

Unfortunately, the principal cannot observe the action taken by each agent. The principal must observe $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, which are correlated with the agents' actions $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, and base the agents' payment on the observed realization of $\mathbf{y}$. This complicates the principal's problem, because they now have to take into account moral hazard.

4.1 Risk Neutrality

We've explicitly stated our assumption that the principal is risk neutral. An attentive reader may have noticed that we have also assumed that the agents are risk averse by our formulation of the utility function $U^i(m_i, a_i)$, specifically by requiring that $d^2u < 0$. Since u is concave and $\mathbf{y}$ is a random variable, we know that

$$E\big[u[R_i(\mathbf{y})]\big] \leq u\big[R_i[E(\mathbf{y})]\big]$$

by Jensen's Inequality, which states that the expected value of a concave function evaluated over a random variable is greater than or equal to the value of the concave function evaluated at the expected value of the random variable, with equality if and only if the function is not strictly concave or the random variable follows a degenerate distribution, i.e. it is in fact a constant. From the above it's clear to see that

$$E\big[u[R_i(\mathbf{y})]\big] - a \leq u\big[R_i[E(\mathbf{y})]\big] - a \quad \forall a,$$

which means that, all else equal, an agent will prefer a constant payment rather than a random payment with the same expected value.

Our assumption that the agents are risk averse not only makes sense from a pragmatic point of view, for the reasons discussed above that economists usually assume the law of diminishing marginal utility, but also from a theoretical one. In the case that the agent is risk neutral, it is easy to see that the principal can achieve an equivalent outcome to the first best by shifting all of the risk to the agent–by "selling the franchise" to the agent. The principal does this by paying the agent

$$I^* = y - k,$$

where k is a constant "franchise fee" equal to the expected surplus resulting from the agent taking the efficient action a_{fb}

$$\begin{aligned} k &= \int y f(y; a_{fb})\,\mathrm{d}y - \gamma(a_{fb} + u^0) \\ &= \int y f(y; a_{fb})\,\mathrm{d}y - C_{FB}(a_{fb}). \end{aligned}$$

Faced with this compensation scheme, the agent chooses an action level a to solve

$$
\begin{aligned}
&\operatorname*{argmax}_a E[u(I^*) - a] \\
=&\operatorname*{argmax}_a E[u(I^*)] - a && \text{(a is constant)} \\
=&\operatorname*{argmax}_a u(E[I^*]) - a && \text{(because agent is risk neutral, } d^2u = 0) \\
=&\operatorname*{argmax}_a u\Big[\int yf(y;a) - yf(y;a_{fb}) + C_{FB}(a_{fb})\,\mathrm{d}y\Big] - a \\
=&\operatorname*{argmax}_a u\Big[\int yf(y;a)\,\mathrm{d}y\Big] - u\Big[\int yf(y;a_{fb})\,\mathrm{d}y\Big] + u\Big[\int C_{FB}(a_{fb})\,\mathrm{d}y\Big] - \gamma(a) \\
& && \text{(because } d^2u = 0 \text{ and } u(0) = 0) \\
=&\operatorname*{argmax}_a \int yf(y;a)\,\mathrm{d}y - \gamma(a). && \text{(dropping constants from maximization)} \\
=&\operatorname*{argmax}_a \int yf(y;a)\,\mathrm{d}y - \gamma(a + u^0)
\end{aligned}
$$

The last step above just increases the objective function by a constant, since we are assuming that the agent is risk neutral and

$$
\begin{aligned}
d^2u = 0 &\iff d^2\gamma = 0 \\
u(0) = 0 &\iff \gamma(0) = 0
\end{aligned}
$$

Note that the final objective function above is the same as the objective function the principal seeks to maximize in the first best case (3.3). The agent's incentives are aligned with the principal's, and the agent will choose to take the efficient level of action a_{fb}. The agent will expect to have a net utility level equal to their reservation utility, while the principal will have a guaranteed profit equal to the profit he would expected in the first best case, P_{FB}.

We've shown that if the agent is not risk averse, the principal can design a payment scheme which achieves the first best outcome. However, when the agent is risk averse such a scheme cannot be used. In the formulation we will examine, the agent's risk aversion pushes the principal away from the compensation scheme outlined above, in which the agent assumes all of the risk of the business. Since the agent is risk averse and the principal is risk neutral, the optimal risk sharing arrangement is that the principal assumes all risk and the agent receives a flat payment. However, such a payment scheme is not possible in the second best case, because the agent has no incentive to choose a high action level if their payment is not dependent on output. In the second best case with risk aversion, then, the principal's optimal payment scheme will lie somewhere between shifting all of the risk onto the agent and assuming all risk themselves by making a constant payment to the agent.

4.2 Contracts

The above compensation scheme is an example of an individual contract, in which each agent's pay depends only on their own level of output. The performance of other agents does not have any effect on the payment. More generally, we can describe a contract as a payment function and a decision rule.

The payment function above was $I^* = y - k$. In the general case, we will write a payment function as $R : \mathbb{R} \to \mathbb{R}_+$ or, in utility terms, $v : \mathbb{R} \to [0, B]$ where, as defined in definition 2.1, $v(y_i) = u(R(y_i))$.

The decision rule is a function, $A : \mathbb{R} \to \mathbb{R}_+$, that relates the signal observed by an agent, σ_i, to the effort level that agent chooses, $A(\sigma_i)$, to maximize their expected level of utility. The factors affecting an agent's choice of effort level are the agent's information about the value of θ, summarized in the distribution G, the reward function facing the agent, and F. In this paper, I focus on symmetric equilibria in which all agents share the same decision rule ex ante, and choose different levels of effort only if they observe different signals about the value of θ. Above, the decision rule was $A(\sigma_i) = a_{fb}\ \forall \sigma_i$, meaning that each agent will take the first best efficient action regardless of what he anticipates θ to be.

Given G, F, and a payment function v, a decision rule A is said to be "valid" if, for all σ_i, $A(\sigma_i)$ is a solution to the Agent's Problem (2.1),

$$\operatorname*{argmax}_a \int v(y) \int f(y - \theta; a)\, \mathrm{d}G(\theta, \sigma_{-i}|\sigma_i)\, \mathrm{d}y - a,$$

where $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$.

With a contract, then, the principal must choose a payment function and a decision rule (v, A) to maximize expected profits (2.2) subject to the constraint that A be a valid decision rule for the agent given v, called the "incentive compatibility constraint", and the constraint that the agent have an ex ante expected utility of at least u^0, called the "participation constraint." The participation constraint is ex ante because it only requires that the agent's expected utility before they observe σ_i is greater than u^0, and not necessarily that, once σ_i is observed, their expected utility is greater than u^0 given σ_i. The incentive compatibility constraint applies ex post the realization of σ_i; $A(\sigma_i)$ must maximize (2.1) for every σ_i.

Definition 4.1. Given G, the joint distribution of θ and σ, define the set of feasible contracts

$$\begin{aligned} S_{ci}(G) \equiv \{(v, A) \;\big|\; & v : (R)_+ \to [0, B], A : \mathbb{R} \to \mathbb{R}_+; \\ & A(\sigma_i) \in \operatorname*{argmax}_a \int v(y) \int f(y - \theta; a)\, \mathrm{d}G(\theta, \sigma_{-i}|\sigma_i)\, \mathrm{d}y - a, \forall \sigma_i; \quad (4.3a) \\ & \iint [v(y) - A(\sigma_i)] f[y - \theta; A(\sigma_i)]\, \mathrm{d}G(\theta, \sigma_i)\, \mathrm{d}y \geq u^0\}. \quad (4.3b) \end{aligned}$$

Here, (4.3a) is the incentive compatibility constraint which must hold in order for A to be a valid decision rule, and (4.3b) is the participation constraint which must hold in order for the agent to agree, ex ante, to the contract rather than simply walking away from the arrangement with his reservation utility u^0.

Note that for any G, we know that $S_{ci}(G)$ is non-empty because it must contain the "no incentive" contract, $(v^0 \equiv u^0, A^0 \equiv 0)$

Definition 4.2. The expected payoff to the principal of a contract $(v, A) \in S_{ci}(G)$ is

$$P_{ci}(v, A, G) \equiv \iint \big(y - \gamma[v(y)]\big) f[y - \theta; A(\sigma_i)]\, \mathrm{d}y\, \mathrm{d}G(\theta, \sigma). \tag{4.4}$$

As an example, the expected payoff of the no incentive contract is

$$\begin{aligned}
&P_{ci}(v^0, A^0, G) && (4.5)\\
=&\iint [z + \theta - \gamma[v^0(y)] f(z; 0)\, \mathrm{d}z\, \mathrm{d}G(\theta, \sigma)\\
=&\iint z f(z; 0)\, \mathrm{d}z\, \mathrm{d}G(\theta, \sigma) + \iint \theta f(z; 0)\, \mathrm{d}z\, \mathrm{d}G(\theta, \sigma) - \iint \gamma(u^0) f(z; 0)\, \mathrm{d}z\, \mathrm{d}G(\theta, \sigma)\\
&&& \text{(from definition, } v^0(y) = u^0)\\
=&\int z f(z; 0)\, \mathrm{d}z \qquad + 0 \qquad - \gamma(u^0) \iint f(z; 0)\, \mathrm{d}z\, \mathrm{d}G(\theta, \sigma)\\
&&& \text{(because } E[\theta] = 0, \text{ and } \theta, z \text{ independent)}\\
=&\int z f(z; 0)\, \mathrm{d}z - \gamma(u^0) \cdot 1\\
&&& \text{(integrating a density function over entire domain)}\\
=&P^0.
\end{aligned}$$

It is interesting to note that the expected payoff from this contract does not vary with G. Its value is always exactly the expected value of production given that the agent takes the least costly action, 0, less the cost of providing the agent with their reservation utility, $\gamma(u^0)$.

4.3 Tournaments

Rather than offering an individual contract (v, A) to each agent in which the agent's compensation is a function of the realized value of y_i, the principal may opt to compensate agents using a tournament payment scheme. In a tournament each agent's compensation is a function of the ordinal ranking of that agent's output relative to the outputs of all other agents. For a n-person tournament, let $\mathbf{W} = (W_1, W_2, \ldots, W_n)$ be a vector of the tournament prizes, so the agent with the highest production receives W_n and the agent with the second-lowest production receives W_2. Further, let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be defined by $w_i \equiv u(W_i)$, $\forall i$.

Define the jth order statistic of a statistical sample to be the jth-smallest value of the sample. For instance, if our sample was $\{1, 3, 8, -4\}$, then the 1st order statistic would be -4, and the 3rd order statistic would be 3.

Since each agent's production is given by $y_i = z_i + \theta$, with θ common across all agents, the rank order of the agents' outputs is independent of the realization of (θ, σ), and depends only on the z_i's. Then, agent i wins prize w_j if and only if z_i is the jth-order statistic of $\{z_1, z_2, \ldots, z_n\}$.

Definition 4.3. Let $\phi_{j,n}(z; a)$ be the density function of the jth order statistic in a sample of size n drawn from the distribution $F(\ ; a)$:

$$\phi_{j,n}(z;a) \equiv \frac{n!}{(n-j)!(j-1)!} f(z;a)[F(z;a)]^{j-1}[1-F(z;a)]^{n-j}.$$

We can verify the accuracy of the above definition as follows. In order for the jth order statistic to be very near a given value, z, we need $j-1$ values in the sample to be less than z. These values can be selected from the n total values in $\binom{n}{j-1}$ ways, since their order is unimportant, and the probability of each of those chosen $j-1$ values being less than z is $F(z; a)$. We also need $n-j$ of the remaining $n-(j-1)$ values to be greater than z. There are $\binom{n-(j-1)}{n-j}$ ways to select these values, and each is greater than z with probability $[1 - F(z; a)]$. Finally, the remaining value must be very near z, which occurs with likelihood $f(z; a)$. Combining all of the above, see see that

$$\begin{aligned} \phi_{j,n}(z;a) &= \binom{n}{j-1}[F(z;a)]^{j-1}\binom{n-(j-1)}{n-j}[1-F(z;a)]^{n-j} f(z;a) \\ &= \frac{n!}{(n-j)!(j-1)!} f(z;a)[F(z;a)]^{j-1}[1-F(z;a)]^{n-j}. \end{aligned}$$

Definition 4.4. Given n and G, define the set of feasible tournaments

$$S_t(n,G) \equiv \Big\{(\mathbf{w},\bar{a}) \;\Big|\; \mathbf{w} \in [0,B]^n, \bar{a} \in \mathbb{R}_+;$$

$$\bar{a} \in \operatorname*{argmax}_a \frac{1}{n}\sum_{j=1}^{n} w_j \int \frac{f(z;a)}{f(z;\bar{a})}\phi_{j,n}(z;\bar{a})\,\mathrm{d}z - a; \tag{4.6a}$$

$$\frac{1}{n}\sum_{j=1}^{n} w_j - \bar{a} \geq u^0\Big\}. \tag{4.6b}$$

Here, the incentive compatibility constraint is (4.6a). The term $\int \frac{f(z;a)}{f(z;\bar{a})}\phi_{j,n}(z;\bar{a})\,\mathrm{d}z$ is the probability that the agent's output is the jth order statistic in the tournament, given that every other tournament participant chooses action $\bar{a}$ and the agent chooses action a. The participation constraint is (4.6b), which requires that the agent's expected utility from participating in the tournament is at least as much as his reservation utility. Similarly to the case for contracts, for any n and G we know $S_t(n, G)$ must be nonempty because it contains at least the "no incentive" tournament $(\mathbf{w}^0 \equiv (u^0, \ldots, u^0), \bar{a}^0 \equiv 0)$.

Definition 4.5. Given n, G, and a tournament $(\mathbf{w}, \bar{a}) \in S_t(n, G)$, let $P_t(n, \mathbf{w}, \bar{a}, G)$ be the principal's expected payoff per agent under tournament $(\mathbf{w}, \bar{a})$

$$\begin{aligned}
P_t(n, \mathbf{w}, \bar{a}, G) &= \iint y f(y-\theta; \bar{a}) \, \mathrm{d}G(\theta, \sigma) \, \mathrm{d}y - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j) \\
&= \iint (z-\theta) f(z; \bar{a}) \, \mathrm{d}G(\theta, \sigma) \, \mathrm{d}z - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j) \\
&= \iint z f(z; \bar{a}) \, \mathrm{d}G(\theta, \sigma) \, \mathrm{d}z - \iint \theta f(z; \bar{a}) \, \mathrm{d}G(\theta, \sigma) \, \mathrm{d}z - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j) \\
&= \int z f(z; \bar{a}) \, \mathrm{d}z - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j). \quad \text{(because } E[\theta] = 0, \text{ and } \theta, z \text{ independent)}
\end{aligned}$$

As an example, the principal's expected payoff under the no incentive tournament is

$$\begin{aligned}
P_t(n, \mathbf{w}^0, \bar{a}^0, G) &= \int z f(z; \bar{a}^0) \, \mathrm{d}z - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j^0) \\
&= \int z f(z; 0) dz - \gamma(u^0) \\
&= P^0.
\end{aligned} \tag{4.7}$$

So the expected payoff to the principal under the no incentive tournament and the no incentive contract are the same, both being equal to the expected production of an agent taking action $a = 0$ minus the cost of providing the agent with exactly their reservation utility, $\gamma(u^0)$.

Note that neither the set of feasible tournaments $S_t(n, G)$ nor the principal's expected payoff for a tournament $P_t(n, \mathbf{w}, \bar{a}, G)$ depend on the distribution G. In light of this, for the rest of the paper we will write the set of feasible tournaments as $S_t(n)$ and the expected payoffs as $P_t(n, \mathbf{w}, \bar{a})$. In addition to simplifying the notation, the independence of the set of feasible contracts and their payoffs from the distribution G will be useful in a later proof.

5 Comparison of Tournaments and Contracts

Whether an individual contract or a tournament should be used in a given situation depends on the variance of the common shock θ. An individual contract is based only on the individual agent's output, so the variance in payment under an individual contract comes from common shock and from the idiosyncratic factors summarized in $F(\ ; a)$. A tournament eliminates the variance in payment due to the common shock factor by using the rank order of an agent's output instead of their gross output to determine payment, but includes variance from not only the individual agent's idiosyncratic factors, but also the idiosyncratic factors of all other agents. Since the agents are risk averse, it is optimal for the variance in payment to be as small as possible ceteris paribus. So, if the variance of the common shock is large enough relative to the variance of each agents' idiosyncratic factors, it makes sense that it would be optimal to use a tournament payment scheme to eliminate the variance from the common shock. However, if the common shock is very small or non-existent, it makes sense that it would be optimal to use a contract payment scheme so that each agent's payment doesn't have variance introduced by other agents' idiosyncratic factors. We will first show that if there is no common error term between the agents, then for any feasible tournament there is a feasible contract that dominates it.

Before we begin the theorem, we need to prove a lemma we will use in the theorem's proof.

Lemma 5.1. *γ is strictly increasing and convex*

Let $x, y \in \mathbb{R}_+$ be given, and let $a = u(x)$, $b = u(y)$. Since u is strictly increasing,

$$\begin{aligned} \gamma(a) < \gamma(b) &\iff u[\gamma(a)] < u[\gamma(b)] \\ u[\gamma(a)] < u[\gamma(b)] &\iff a < b, \qquad (\gamma = u^{-1}) \end{aligned}$$

so,

$$\gamma(a) < \gamma(b) \iff a < b$$

and γ is strictly increasing.

Since u is concave,

$$\begin{aligned} &\forall\, \alpha > 0, \beta > 0, \alpha + \beta = 1, \\ &u(\alpha x + \beta y) \geq \alpha u(x) + \beta u(y). \end{aligned}$$

Since γ is strictly increasing,

$$\begin{aligned} \gamma[u(\alpha x + \beta y)] &\geq \gamma[\alpha u(x) + \beta u(y)] \\ \alpha x + \beta y &\geq \gamma(\alpha a + \beta b) \\ \alpha\gamma(a) + \beta\gamma(b) &\geq \gamma(\alpha a + \beta b), \end{aligned}$$

so γ is convex. $\square$

Theorem 5.2. *Given G, F, $n \geq 2$, if $\theta = 0$, i.e. if*

$$\int_{\sigma \in \mathbb{R}^n} dG(\theta, \sigma) = \begin{cases} 0 & \text{if } \theta < 0, \\ 1 & \text{if } \theta \geq 0, \end{cases} \tag{5.8}$$

then for any feasible tournament $(\mathbf{w}, \bar{a}) \in S_t(n)$, there is a feasible contract $(v, A) \in S_{ci}(G), i = 1, \ldots, n$ such that

$$P_{ci}(v, A, G) \geq P_t(n, \mathbf{w}, \bar{a}) \quad i = 1, \ldots, n,$$

with equality if and only if $(\mathbf{w}, \bar{a}) = [(u^0, \ldots, u^0), 0]$, the no incentive tournament.

Given G, F, $n \geq 2$, $(\mathbf{w}, \bar{a}) \in S_t(n)$, let (v^*, A^*) be a contract defined by

$$\begin{aligned} v^*(y) &= \frac{1}{n} \sum_{j=1}^{n} w_j \frac{\phi_{j,n}(y; \bar{a})}{f(y; \bar{a})} \\ &= \frac{1}{n} \sum_{j=1}^{n} w_j \frac{n!}{(n-j)!(j-1)!} [F(z; \bar{a})]^{j-1} [1 - F(z; \bar{a})]^{n-j} \quad \forall y, \end{aligned}$$

$$A^*(\sigma_i) = \bar{a} \quad \forall \sigma_i.$$

This contract assigns (in utility terms) a pay-off to an agent with production y the expected value of the payoff that the agent would have received for producing y, given that all agents choose action $\bar{a}$ in the tournament $(\mathbf{w}, \bar{a})$. I will now show that this contract satisfies the contract incentive compatibility constraint (4.3a) and the contract participation constraint (4.3b).

An agent facing contract (v^*, A^*) aims to maximize his expected utility given σ_i, and so will choose an action $A^*(\sigma_i)$ such that

$$\begin{aligned} A^*(\sigma_i) &\in \operatorname*{argmax}_a \int v^*(y) \int f(y - \theta; a) \, \mathrm{d}G(\theta, \sigma_{-i} | \sigma_i) \, \mathrm{d}y - a \\ &= \operatorname*{argmax}_a \int v^*(y) f(y; a) \, \mathrm{d}y - a, \end{aligned}$$

where the simplification follows from (5.8) above. Substituting in the reward function for $v^*(y)$, we can see

$$\begin{aligned} &= \operatorname*{argmax}_a \int \frac{1}{n} \sum_{j=1}^{n} \left[w_j \frac{\phi_{j,n}(y; \bar{a})}{f(y; \bar{a})} \right] f(y; a) \, \mathrm{d}y - a \\ &= \operatorname*{argmax}_a \frac{1}{n} \sum_{j=1}^{n} w_j \int \frac{f(y; a)}{f(y; \bar{a})} \phi_{j,n}(y; \bar{a}) \, \mathrm{d}y - a. \end{aligned}$$

Since $(\mathbf{w}, \bar{a})$, satisfies the tournament incentive compatibility constraint (4.6a), it follows that $\bar{a} \in A^*(\sigma_i)$ and so (v^*, A^*) satisfies the contract incentive compatibility constraint (4.3a).

To show our contract satisfies the participation constraint (4.3b) we examine the expected utility of each agent, which is

$$
\begin{aligned}
&\int v^*(y) \int f(y-\theta;\bar{a})\,\mathrm{d}G(\theta,\sigma_{-i}|\sigma_i)\,\mathrm{d}y - \bar{a} \\
=&\int v^*(y) f(y;\bar{a})\,\mathrm{d}y - \bar{a} && \text{(by (5.8))} \\
=&\int \Big[\frac{1}{n}\sum_{j=1}^{n} w_j \frac{\phi_{j,n}(y;\bar{a})}{f(y;\bar{a})}\Big] f(y;\bar{a})\,\mathrm{d}y - \bar{a} && \text{(by definition of } v^*\text{)} \\
=&\frac{1}{n}\sum_{j=1}^{n} w_j \int \frac{f(y;\bar{a})}{f(y;\bar{a})}\phi_{j,n}(y;\bar{a})\,\mathrm{d}y - \bar{a} && \text{(rearranging terms)} \\
=&\frac{1}{n}\sum_{j=1}^{n} w_j \int \phi_{j,n}(y;\bar{a})\,\mathrm{d}y - \bar{a} \\
=&\frac{1}{n}\sum_{j=1}^{n} w_j - \bar{a} && \text{(because } \phi_{j,n}(\ ;\bar{a}) \text{ is a density function)} \\
\geq&\ u^0.
\end{aligned}
$$

The inequality results from the assumption that the tournament $(\mathbf{w}, \bar{a})$ satisfies the tournament participation constraint (4.6b). Thus, the contract (v^*, A^*) satisfies the contract participation constraint (4.3b).

All that remains is to show that our constructed contract weakly dominates the given tournament. We will do so using Jensen's Inequality and the fact that $\gamma = u^{-1}$ is strictly convex. The principal's expected payoff from contract (v^*, A^*) is (from (4.4))

$$
\begin{aligned}
P_{ci}(v^*, A^*, G) &= \iint \{y - \gamma[v^*(y)]\} f[y-\theta; A^*(\sigma_i)]\,\mathrm{d}y\,\mathrm{d}G(\theta,\sigma) \\
&= \int \{y - \gamma[v^*(y)]\} f[y; A^*(\sigma_i)]\,\mathrm{d}y && \text{(by (5.8))} \\
&= \int \Big\{ y - \gamma\Big[\frac{1}{n}\sum_{j=1}^{n} w_j \frac{\phi_{j,n}(y;\bar{a})}{f(y;\bar{a})}\Big]\Big\} f(y;\bar{a})\,\mathrm{d}y \\
&= \int y f(y;\bar{a})\,\mathrm{d}y - \int \gamma\Big[\frac{1}{n}\sum_{j=1}^{n} w_j \frac{\phi_{j,n}(y;\bar{a})}{f(y;\bar{a})}\Big] f(y;\bar{a})\,\mathrm{d}y \\
&\geq \int y f(y;\bar{a})\,\mathrm{d}y - \int \Big[\frac{1}{n}\sum_{j=1}^{n} \gamma(w_j) \frac{\phi_{j,n}(y;\bar{a})}{f(y;\bar{a})}\Big] f(y;\bar{a})\,\mathrm{d}y \\
& && \text{(by Jensen's Inequality and Lemma 5.1)} \\
&= \int y f(y;\bar{a})\,\mathrm{d}y - \frac{1}{n}\sum_{j=1}^{n} \gamma[w_j] \int \phi_{j,n}(y;\bar{a})\,\mathrm{d}y \\
& && \text{(canceling } f(y:\bar{a}) \text{ and rearranging)} \\
&= \int y f(y;\bar{a})\,\mathrm{d}y - \frac{1}{n}\sum_{j=1}^{n} \gamma(w_j) \\
& && \text{(because } \phi_{j,n}(y;\bar{a}) \text{ is a density function)} \\
&= P_t(n, \mathbf{w}, \bar{a}) \\
& && \text{(by Definition 4.5)} \\
&\therefore P_{ci}(v^*, A^*, G) \geq P_t(n, \mathbf{w}, \bar{a}).
\end{aligned}
$$

The inequality is strict unless $\mathbf{w} = (\bar{w}, \dots, \bar{w})$. If $\mathbf{w} = (\bar{w}, \dots, \bar{w})$, then we must have $\bar{a} = 0$ since agents have no incentive to take higher action levels because the first and last place are paid the same. Further, unless $\mathbf{w} = (\bar{w}, \dots, \bar{w}) = (u^0, \dots, u^0)$, the no incentive contract (v^0, A^0) dominates $(\mathbf{w}, \bar{a})$. So, the inequality is strict unless $(\mathbf{w}, \bar{a})$ is the no incentive tournament. □

In situations with no common error term, individual contracts should be preferred to the use of tournaments. When the $\theta = 0$ there isn't any "noise" common to all agents, so there isn't any uncertainty about each agent's level of effort that can be removed by comparing agents' productions to one another. Instead of removing common "noise", by using a tournament we are introducing more variation because each agents' compensation has variance from not only their own z_i distribution, but that of all other agents as well. Since the agents are risk averse, in the absence of a shared error term this added volatility leaves them worse off under a tournament than they would be under a individual contract.

However, if there is a common error term and the distribution of that error term is sufficiently diffuse, we will show that it is optimal to use tournaments instead of contracts to determine the pay received by each agent. Let $\{G_k\}_{k=1}^{\infty} = \{G_1, G_2, \ldots\}$ be a sequence such that for all k,

$$\begin{gathered} G_k \text{ has a density function } g_k, \text{ and} \\ \int_{\sigma_{-i} \in \mathbb{R}^{n-1}} g_k(\theta, \sigma_{-i} | \sigma_i) \, \mathrm{d}\sigma_{-i} \equiv g_{ki}(\theta|\sigma_i) < 1/k \quad \forall\, \theta, \sigma_i, i. \end{gathered} \tag{5.9}$$

Here, (5.9), means that as k becomes large, the common shock is extremely variable, and for each agent i the density function g_{ki} is very small everywhere. We will now show that given a sequence $\{G_k\}_{k=1}^{\infty}$ as defined above, for sufficiently large k the optimal contract is the "no incentive" contract (v^0, A^0), with payoff P^0. As shown above, the optimal tournament does not depend on the distribution G_k, so the principal's payoff under the optimal tournament is at least P^0.

Theorem 5.3. *Let $n \geq 2$, F, and $\{G_k\}_{k=1}^{\infty}$ satisfying* (5.9) *be given. If $F_a(z;a)$ is of bounded variation in z, and the bound, M, is uniform in a for $a > 0$, then there exists K such that,*

$$\forall\, k > K, \quad \max_{(\mathbf{w}, \bar{a}) \in S_t(n)} P_t(\mathbf{w}, \bar{a}, n) \geq \max_{(v,A) \in S_{ci}(G_k)} P_{ci}(v, A, G_k), \;\; i = 1, \ldots, n.$$

with equality if and only if $\max_{(\mathbf{w}, \bar{a}) \in S_t(n)} P_t(\mathbf{w}, \bar{a}, n) = P^0$*, the payoff from the no incentive tournament.*

Let $\{(v_k, A_k)\}_{k=1}^{\infty}$ be a sequence of optimal contracts under the given conditions. We know by (4.3a) that each agent's choice of action must be a solution to the maximization problem

$$\operatorname*{argmax}_a \int v(y) \int f(y-\theta; a) g_{ki}(\theta|\sigma_i) \, \mathrm{d}\theta \, \mathrm{d}y - a.$$

Now, if for any agent i and any signal σ_i, we have $A_k(\sigma_i) > 0$, then we know that $A_k(\sigma_i)$ is an interior solution to the above maximization and the first derivative of the objective function with respect to a is 0. So, if $A_k(\sigma_i) > 0$, then we must have

$$\begin{aligned} 0 &= \frac{\partial}{\partial a}\left[\int v(y) \int f(y-\theta; a) g_{ki}(\theta|\sigma_i) \, \mathrm{d}\theta \, \mathrm{d}y - a \right] \\ \Rightarrow 1 &= \frac{\partial}{\partial a}\left[\int v(y) \int f(y-\theta; a) g_{ki}(\theta|\sigma_i) \, \mathrm{d}\theta \, \mathrm{d}y \right] \\ \Rightarrow 1 &= \int v(y) \int f_a(y-\theta; a) g_{ki}(\theta|\sigma_i) \, \mathrm{d}\theta \, \mathrm{d}y \end{aligned}$$

However, we know that

$$\begin{aligned}
&\lim_{k\to\infty} \left| \int f_a(y-\theta;a) g_{ki}(\theta|\sigma_i)\,\mathrm{d}\theta \right| \\
&\le \lim_{k\to\infty} \frac{1}{k} \int |f_a(y-\theta;a)|\,\mathrm{d}\theta && \text{(by (5.9))} \\
&\le \lim_{k\to\infty} \frac{M}{k} && \text{(because } F_a(z;a) \text{ is of bounded variation)} \\
&= 0.
\end{aligned}$$

Taking the above result and the fact that the range of $v(y)$ is $[0, B]$, it apparent that for sufficiently large K,

$$1 > \int v(y) \int f_a(y-\theta;a) g_{ki}(\theta|\sigma_i)\,\mathrm{d}\theta\,\mathrm{d}y, \ \forall\, k > K.$$

Which implies that for $k > K$, $A_k(\sigma_i) = 0$ for all σ_i, and every contract in $\{(v_k, A_k)\}_{k=K}^{\infty}$ is the no incentive contract (v^0, A^0). As shown in (4.5) and (4.7), the expected payoff of the no incentive contract and the no incentive tournament are both equal to

$$P^0 = \int y f(y;0)\,\mathrm{d}y - \gamma(u^0).$$

Further, as noted at the end of Section 4.3, neither the optimal tournament nor the optimal tournament's payoff depend on G, so the payoff of the optimal tournament does not change as $k \to \infty$, even as the payoff of the optimal contract goes to P^0. So, we have

$$\forall\, k > K, \ \max_{(\mathbf{w},\bar{a}) \in S_t(n)} P_t(\mathbf{w}, \bar{a}, n) \ \ge \ P^0 \ = \max_{(v,A) \in S_{ci}(G_k)} P_{ci}(v, A, G_k), \ i = 1, \ldots, n,$$

with equality if and only if $\max_{(\mathbf{w},\bar{a}) \in S_t(n)} P_t(\mathbf{w}, \bar{a}, n) = P^0$. $\square$

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