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ABSTRACT

We study a novel model of random graph which exhibits the structural characteristics of the Watts-Strogatz small-world network. The small-world network is characterized by a high level of local clustering while also having a relatively small graph diameter. The same behavior that makes the Watts-Strogatz model behave like this also makes it difficult to analyze. Our model addresses this issue, closely mimicking the same structure experimentally while following a constructive process that makes it easier to analyze mathematically. We present a bound on the average shortest path length in our new model, which we approach by looking at the two key geometric components.

I. INTRODUCTION

Structurally complex networks arise in many areas of the natural, social, and technological sciences. We can study such networks with random graph models that capture some of their characteristic properties [8], [13]. In both classical random graph theory and in the analysis of real-world networks, we are often interested in the number and size of connected components, the average distance between vertices, and the diameter, the longest finite distance between two vertices in the graph.The classical Erdős-Rényi random graph, or the $G(n, p(n))$, introduces an edge between each pair of its n vertices with uniform probability $p(n)$. Though the connectivity and the diameter change dramatically over the full range of $p(n)$, the $G(n, p)$ does not admit a very complex structure within its connected components.

We are interested in random graph models that are highly connected, exhibit local clustering, yet have a relatively small diameter. By clustering, we mean that the neighbors of a vertex are likely to be connected to each other. Heuristically, the high clustering and low diameter make it easy to reach most other vertices from any starting vertex in the graph. We say networks like these demonstrate the "small-world" phenomenon. Many real-world networks have been shown to be small-world-like such as social networks, where vertices are people and edges exist between people who are acquainted, the internet, and the citation network of mathematical research papers, to name a few. Several random graph models that mimic the small-world phenomenon have been introduced and studied, namely the Watts-Strogatz [13] and the Newman-Watts [9], [10] models. Both of these models are based on a configuration of vertices into a k-regular ring lattice, where each vertex has an edge that connects it to its k nearest consecutive vertices (mod n). Then a set of random edges is introduced in some way that reduces average shortest path length and diameter.

In this paper, we introduce a novel random graph model we call the *railroad model* that is another variation of a small-world network. We will study the average shortest path length in the railroad model in the context of prior work. We use classical approaches to the theory of random graphs from [11], [6], [2], [4], and interpolate original results with established results on the diameter of the $G(n, p)$ from [12], using the railroad model as a lens through which to study the same properties in the Watts-Strogatz model, about which less is known.

In the railroad model we begin with n^2 vertices (for convenience later) configured into a regular ring-lattice structure, and then select a subset of $2n$ vertices we will call *train stations*. We then introduce asymptotically λn random edges between the train stations which we will call *rail lines* or *shortcuts*. After we add the shortcuts, we remove the same number of edges from the outer lattice to maintain the original number of edges in the graph. This way, the railroad model has asymptotically the same number of edges as in the Watts-Strogatz regime for the same edge-density $p(n)$, so that the models are more comparable. We focus on the range of $p(n)$ where $p = \lambda/(2n)$, $\lambda > 1$ fixed, and analyze the average path length in the railroad model, leading toward our conjectured main result.

A. Main Conjecture

Let G^* be a graph on n^2 vertices drawn from the railroad random graph model. Let $\mathcal{L}$ be the average shortest path length over all pairs of vertices in G^*. We conjecture that the expected value of $\mathcal{L}$ is given by

$$\mathbb{E}[\mathcal{L}(G^*)] \leq \frac{1}{2} n \log n + O(\log_\lambda 2n). \tag{1}$$

While we could write the above result as $O(\frac{1}{2} n \log n)$, without the extra $O(\log_\lambda 2n)$ term, we choose to give the secondary term specifically to look at the intricacies that come from the two geometric components of the railroad model's structure.

In Section 2, we introduce the existing random graph models that are relevant so their results may be referenced later, and then introduce the railroad model in more detail and explore the behavior that motivates the main conjecture in (1), and in Section 3 we give the detailed proofs to support (1). Along the way, we observe both experimentally and analytically that the average path length in the railroad model is very close to that of the Watts-Strogatz, supporting our use of the railroad model as a simpler method to study the same network properties.

II. THE RAILROAD MODEL

A. Relevant Random Graph Models

1. The Erdős-Rényi random graph

The Erdős-Rényi is the simplest of random graph models, We will refer to the $G_{n,p}$ and the Erdős-Rényi network interchangeably, where the classical results can be found in [7], [4] and [11]. In the $G_(n,p)$, for each of the $\binom{n}{2}$ pairs of vertices, an edge between them is present independently with probability $p(n)$, the *edge density*. Consequently, the expected number of edges in the $G_{n,p}$ is $p\binom{n}{2} = \Theta(pn^2/2)$ for large n.

The edge density $p(n)$ can take on a range of values, either staying fixed or varying with n. $p(n)$ determines fundamental characteristics of the graph such as the number and structure of connected components and the *diameter*. In the range where $p = \lambda/n$, λ fixed, classical results show the $G_{n,p}$ undergoes a phase transition at $\lambda = 1$. When $\lambda < 1$, the $G_{n,p}$ has mostly small disconnected components. When $\lambda > 1$, with probability going to 1 there is a giant component containing most of the vertices, with all other components having $O(\log n)$ vertices. The second variant of the Erdős-Rényi random graph model has a fixed number of edges m introduced in the graph. The results on the $G_{n,m}$ are parallel to those of the $G_{n,p}$, but the proofs are slightly more involved to account for the lack of total independence.

In this paper we will focus on the giant component range of the the $G(n,p)$, where $p(n) = \lambda/n$, $\lambda > 1$ fixed, for our investigation of the railroad model. We use the result from [12] that in this range, the diameter of the $G_{n,p}$ is $\Theta(\log_\lambda n)$.

The disadvantage of the $G(n,p)$ is that the total independence that makes simple analysis possible is also responsible for relatively simple network structure. In the λ/n regime, the $G_{n,p}$ is locally tree-like; the existence of cycles of small length is rare. Various random graph models have since been introduced to mimic the small-world phenomenon, though we will have to give up the computational simplicity when we move toward modeling more complex structures.

2. The Watts-Strogatz small-world network

The first small-world network model was introduced in [13], motivated by an interest in introducing more complex structure into random graphs to reflect natural phenomena in biological, technological, and social networks. These real-world networks exhibit high clustering among local neighborhoods of vertices, yet short average path length globally. The Watts-Strogatz (WS) random graph follows a constructive process where we begin with n vertices arranged in a thickened cycle we define bellow, and randomness is introduced by "rewiring" some of those edges, keeping one end of each edge fixed and redirecting the other end with probability $p(n)$ to a randomly chosen vertex somewhere else in the ring. The resulting graph maintains most of its local clustering, while introducing "shortcuts" that dramatically reduce the average path length and diameter.

Definition. We define a *thickened cycle* with thickness k to be a configuration of n vertices where the vertex set is arranged in a circle, and each vertex v_i is connected to its nearest k neighbors on either side, $\{v_{i-k}, \ldots, v_{i-1}, v_{i+1}, \ldots v_{i+k}\}$, mod n.

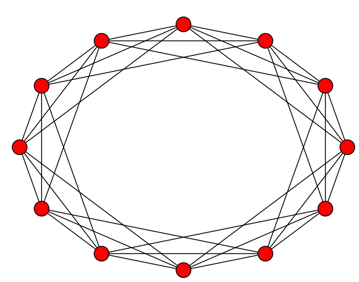


FIG. 1. A thickened cycle with $k = 3$.

To construct a graph W from the Watts-Strogatz model, we begin with n vertices configured into a thickened cycle with thickness k so there are exactly kn edges. For each of the kn edges, we leave one end fixed, and with probability $p(n)$ "rewire" the other end to a different vertex chosen at random from the other $n-2$ vertices in W to form *shortcuts*. We expect that there will be pkn shortcuts in W, included in the total number of edges kn.

The Watts-Strogatz model captures the small-world phenomena, but the process is difficult to analyze–we lose the computational simplicity by following a rewiring procedure rather than introducing shortcuts independently. As a result, much of the work on small-world networks has looked at variants that are easier to analyze, such as the Newman-Watts model.

3. *The Newman-Watts model*

The Newman-Watts random graph model studied in [9],[10], [1] begins with the same thickened cycle as a base network, but instead of rewiring edges, superimposes a $G_{n,p}$ on top of the cycle, where each pair of vertices that is not already connected in the cycle has an edge between them independently with probability p. The Newman-Watts also exhibits the small-world properties of high clustering and low average path length, yet not as difficult to analyze as the Watts-Strogatz because it introduces its random edges by a more independent process. It also is related to the cycle with a random matching, which has been studied extensively in [5].

A key difference stands out between the Newman-Watts and the Watts-Strogatz, because of the heavy amount of random edges added in the superimposed $G(n,p)$, the Newman-Watts has considerably more long-range edges than the Watts-Strogatz. Recall that the $G_{n,p}$ has $\Theta(pn^2/2)$ edges, and the base thickened cycle has kn edges. Thus, a graph G in the Newman-Watts model with $p=\lambda/n$ has $\Theta(\lambda n/2)+kn$ edges while the Watts-Strogatz in the same $p(n)$ range has exactly kn.

B. Constructing the Railroad Random Graph

We are interested in the result that the Watts-Strogatz admits small-world structure *without* having to introduce too many extra shortcuts, and just by rewiring only a handful of edges taken from the outer lattice. This motivates our investigation of the railroad model that mimics the small-world structure exhibited by the Watts-Strogatz more closely than the Newman-Watts does, while still being easier to analyze.

For convenience later, we will construct a graph G^* from the railroad model on n^2 vertices, configured into a baseline thickened cycle. In this paper, we focus on a thickness of $k=2$, and will eliminate k from our notation unless it is necessary for clarity.

We construct a graph G^* from the railroad model as follows. Let G be the thickened cycle of n^2 vertices and $k = 2$. Note that G has $2n^2$ edges and we will maintain this amount in G^*. From the vertex set of G, select a random subset T of $2n$ distinct vertices we will call *train stations*. For every pair of train stations v, v' in T, introduce an edge with probability $\lambda/2n$, with $\lambda > 1$ fixed. An edge is a *railline* if both its ends are in T. Call the resulting graph G', the thickened cycle G combined with the Erdős-Rényi graph on T.

The expected number of raillines is given by

$$\binom{2n}{2}\frac{\lambda}{2n} \approx \frac{(2n)^2}{2}\frac{\lambda}{2n} = \lambda n,$$

asymptotically, as $n \to \infty$.

To keep the number of edges at $2n^2$, we select a subset of $\lfloor \lambda n \rfloor$ edges in the thickened cycle G to remove. Remove these, and call this resulting graph G^*. We prove in section 3.1 that in this range of $p(n)$, the probability that the thickened cycle will remain connected after we remove edges goes to 1.

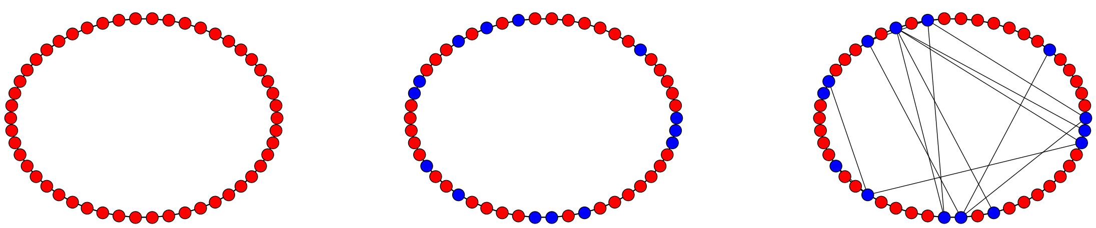


FIG. 2. The stages of the railroad model, G, G', and G^* from left to right, with $\lceil n^2 \rceil = 50$, $2n = 14$ train stations in blue, rail line shortcuts introduced with probability $\lambda/2n = .15$. We cannot see the deleted edges from the outer cycle, but they are there.

Thus, G^* approximately keeps the same number $2n^2$ edges as the thickened cycle, with λn of them being randomized raillines that run through the interior of the graph. Recall that in the Watts-Strogatz, if we were to construct a graph on n^2 vertices with $p = \lambda/(2n)$, we also expect to see $pkn^2 = \lambda n$ shortcuts. Notice that the construction process of G^* consists of two rounds of independent edges introduction and deletion processes, in contrast to the more analytically complicated rewiring process of the Watts-Strogatz, yet the resulting models still appear to "look" similar, see Fig. 3.

The key difference between the railroad model and the Watts-Strogatz is the lack of consistency in the distribution of the endpoints of the randomized edges. In the Watts-Strogatz, edges are essentially deleted from G and added to G' in one go—one end of each

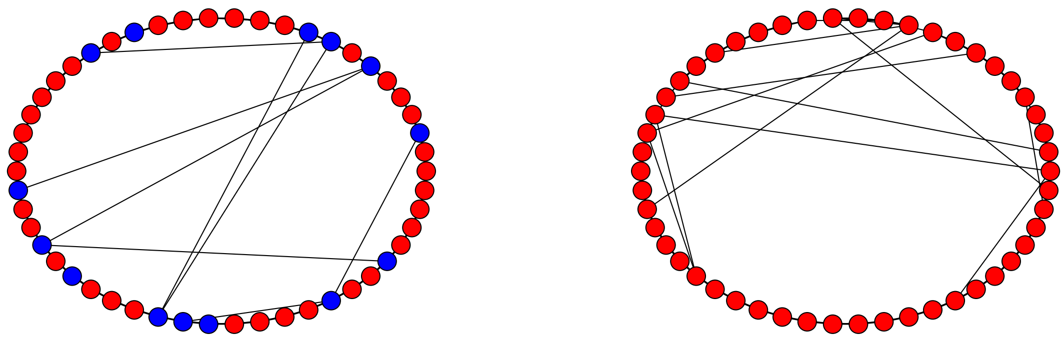


FIG. 3. Left: instance of the railroad model, right: an instance of the Watts-Strogatz, both with $n = 50$, $k = 2$, $\lambda = 1.1 \Rightarrow p \approx .08$.

edge remains fixed, and the other end is rewired with probability $\lambda/2n$. Whereas, in the trains model, an edge may be introduced between two vertices $t, t' \in T$, but all of t and t's neighbors may remain in place in the outer cycle, if edges are removed elsewhere to reach the quota. Nonetheless, this difference should not have a significant effect on the behavior of the railroad model as a comparable alternative to the Watts-Strogatz.

Consider a vertex v in the Watts-Strogatz random graph, supposing that all the rewired edges are firmly in place. We know there are about λn rewired edges. The probability that v is on either end of one of these rewired edges is $2(\lambda n)/n^2 = 2\lambda/n$, the 2 is there because v could be either end. Let v' be in a graph in the railroad model. The probability that v' is the end of a shortcut is first directly related to the probability that $v' \in T$, which is asymptotically $2/n$, and then v' has on average λ edges to other vertices in T. So, the probability that v' is the end of one or more shortcuts is also $2\lambda/n$, providing the heuristic assurance that the models are actually comparable.

C. Experimental Results

We support our discussion of the similarities between the two models with numerical simulations in the same range of $p(n) = \lambda/2n$, with $\lambda > 1$ fixed.

1. Methods:

For the Watts-Strogatz simulations, we configure a thickened cycle, then for each edge we check if it has already been given the chance to rewire, if not, we select a non-neighbor vertex uniformly from the vertex set and rewire the edge there with probability $\lambda/2n$.

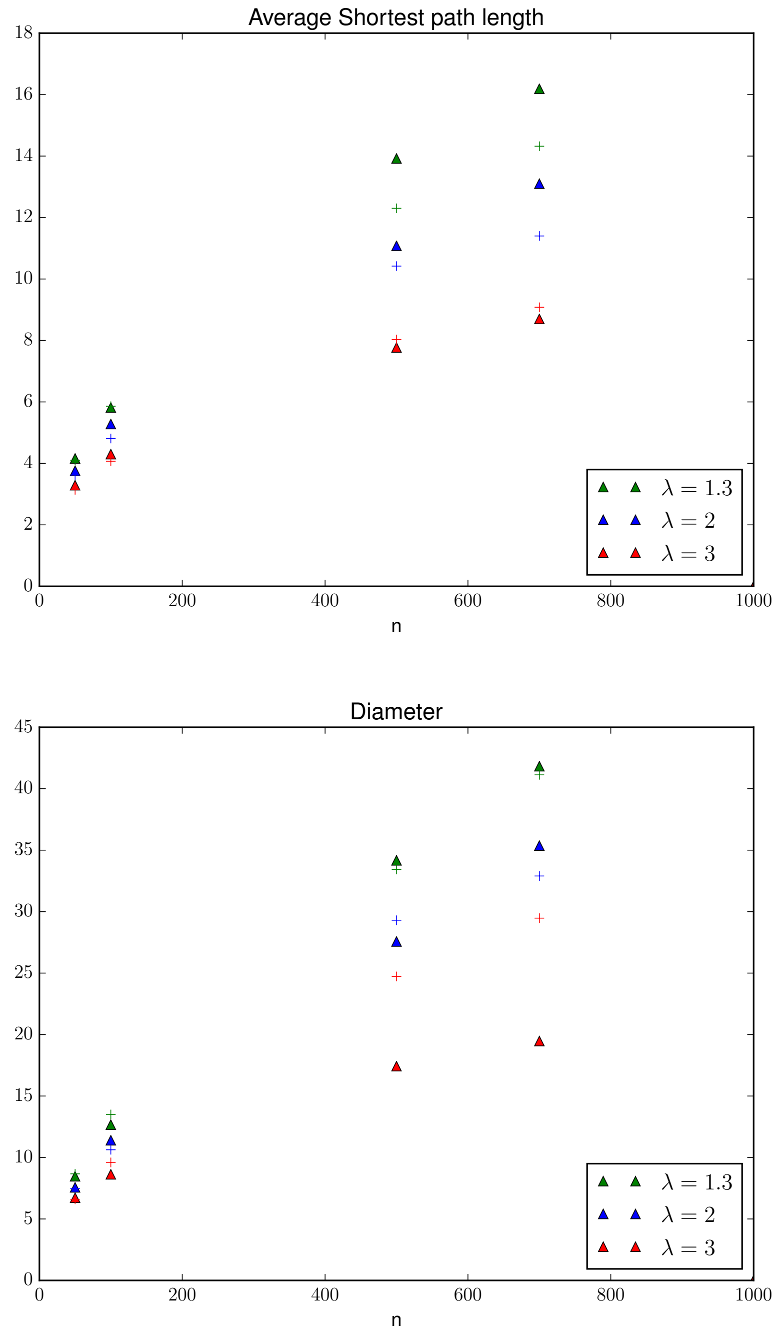


FIG. 4. Comparing the Comparative model ($\triangle$) to the Watts-Strogatz (+) where each marker is the average of 50 simulations. Each color corresponds to the same value of λ for both the + and $\triangle$ models.

In the railroad model, we follow the heuristic honestly, picking $2n$ train station vertices independently without replacement, then place an edge between each pair of stations with probability $\lambda/2n$. We then consider each edge in the outer cycle and remove it with probability $\lambda/2n$ until $\lfloor \lambda n \rfloor$ edges have been removed.

The $+$ markers are the results of the Watts-Strogatz simulations, the $\triangle$ markers are the railroad simulations. We keep λ small to correspond to our range of n. For each instance of the graph, we compute the average path length (and diameter) in the entire graph using the Python NetworkX module. Each symbol represents the mean of those results over 50 simulated graph instances.

D. Main Conjecture on the Average Shortest Path Length

Now that we understand the structure and behavior of the railroad model, we motivate our main conjecture on the average shortest path length with a geometric argument. Say we are interested in finding the shortest distance between some two vertices, $v, v' \in G^*$, selected arbitrarily. Say we begin at v and want to travel to v'. Naturally, we would check to see if v is a train station, i.e. if $v \in T$, which would give us direct access to shortcuts in the rail lines component of the graph. If v is not in T, we would walk along the outer cycle until we reach such a train station, call it w. Simultaneously, we check if $v' \in T$; again, if v' is not, we find the nearest train station $w' \in T$ to v' along the outer cycle.

In this way, the major part of the traversal is before and after utilizing the rail lines, traveling along the outer cycle. Since the train station vertices are distributed randomly around the lattice, we seek to bound the maximum gap between such vertices, which will naturally also bound the maximum graph distance from any vertex to the nearest train station. Since we walk at most half the maximum distance between train stations from v to w and also from v' to w', our bound is honestly given by the expected maximum distance between train stations. The smaller order term in our bound is the average length of a route from one train station to another. Thus, our bound is expressed in terms of two terms bounding the two structural components of the railroad model.

Main Conjecture.

Let G^* be a random graph in the railroad model with n^2 vertices, $2n$ train stations, and edge-density $p = \lambda/2n$. *Let $\mathcal{L}(G^*)$ be the average distance between any two distinct vertices in G^*. We conjecture that the expected value of $L(G^*)$ is*

$$\mathbb{E}\mathcal{L}(G^*) \leq \frac{1}{2} n \log n + O(\log_\lambda 2n).$$

The average distance between vertices in a thickened k-cycle, $k = 2$, with n^2 vertices is $n^2/8$ since we have most of our span-2 edges in place. With our train stations in place, we can cut this distance down. We do so by investigating the spacing of the train station vertices along the outer cycle, and find that the probability that two train stations are more than $\frac{1}{2}n \log n$ apart goes to 0 as $n \to \infty$. We prove our bound in section 3.2.

We note that we make a simplifying assumption in this regime where $k = 2$. We approach bounding the maximum gap between train stations by bounding the number of consecutive vertices between one train station and the next. To formalize the final bound, we need to bound the graph distance, which will be bounded below by half of what we find, since most but not all of the span-2 edges will be in place in the thickened cycle even after deleting λn edges. We make this assumption safely since we prove in Section 3.1 that the cycle will not become disconnected. Consequently, the maximum number of consecutive vertices between train stations will be at most twice the graph distance we are interested in, not affecting the asymptotics by more than a factor of 2. The question is more interesting for $k > 2$, which we do not study in this paper but we keep in mind for future exploration.

III. PROOFS OF THE MAIN RESULT

In this section we give the proofs of the components that work toward the main conjecture in (1) and Section 2.4.

A. Keeping the Outer Cycle Connected

Recall that a *thickened cycle* with thickness k is a configuration of n vertices where and each vertex v_i is connected to its nearest k neighbors on either side, $\{v_{i-k}, \ldots, v_{i-1}, v_{i+1}, \ldots v_{i+k}\}$, mod n. All the models we discussed so far have been based on the thickened cycle structure. We present a range of $p(n)$ for which, as $n \to \infty$, the probability that the cycle becomes disconnected during any of the rewiring (for the Watts-Strogatz) or edge deletion (in the railroad) processes goes to 0. For the following results, we consider an edge to be removed from the cycle if it has been rewired. Say a thickened cycle has been *pruned* if it has had a nonempty subset of its edges removed.

Definition. We say a pruned thickened k-cycle is *intact* if there exists a cycle of length at least n/k, where we can list the vertices in monotonic order mod n.

Definition. Say that two vertices v_i and v_j are *separated* if there is no path from v_i to v_j of length $\leq n/(2k)$ using edges from the original outer k-cycle.

Theorem III.1. *Let W be a WS graph with n nodes and rewiring probability p. Let $m = \frac{k(k+1)}{2}$. Then as long as $p = o(n^{1/m})$, the outer thickened k-cycle will remain intact with probability going to 1 as $n \to \infty$.*

We prove the theorem using the following key lemmas:

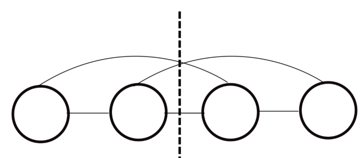
Lemma III.2. *In order to break the k-cycle, an edge between two consecutive vertices v_i and v_{i+1} must be removed.*

In order to separate a pair of vertices v_i and v_k we need to remove edges from the graph so that there is no short path from v_i to v_k. In order to do so, there must be some j such that $i \leq j < j+1 \leq k$ where we remove the edge between v_j and v_{j+1}.

Lemma III.3. *There is a "critical cut" to separate any two consecutive vertices by removing the minimum number of edges possible, and the that minimum is given by $m(k) = \frac{k(k+1)}{2}$ where k is the thickness of the cycle.*

Proof. We proceed by induction on k. When $k = 1$, we can separate v_1 and v_2 if and only if we remove the single edge (v_1, v_2). This is the critical cut, trivially, and satisfies $k(k+1)/2 = 1$.

We motivate the induction by showing what happens when $k = 2$. Again let us look at separating v_1 and v_2. As in the case of a $k = 1$ cycle, we must remove the edge between v_1 and v_2. From the geometry, it is clear that we must also remove the two span-2 edges that cross over the imaginary midline between v_1 and v_2, the edges (v_0, v_2) and (v_1, v_3). Notice that each edge corresponds to one of the two equivalence class mod 2. This is the simplest way we can achieve separating v_1 and v_2, so the critical cut has 3 edges.



We continue with the induction. Assume up to $k-1$ that the number of edges in the critical cut for a $k-1$ cycle is $m(k-1) = (k-1)k/2$. Consider the k-cycle. We can build it by adding span-k edges to a $k-1$ cycle. To separate v_1 and v_2 again with the fewest number of edges, we utilize the critical cut from the $k-1$ cycle, which by virtue of the inductive process is the minimum. Now we address the span-k edges. As in the example with $k = 2$, the critical cut requires that we remove the k closest span-k edges that "cross over" the imaginary midline between v_1 and v_2. There are k such edges, each of which contain two representatives of each equivalence class mod k.

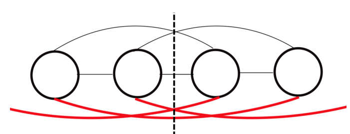


FIG. 5. We add span-3 edges to a $k = 2$ thickened cycle to form a $k = 3$ thickened cycle.

Thus, the number of edges in the critical cut for a k cycle is

$$m(k-1)+k=\frac{(k-1)k}{2}+k=\sum_{j=1}^{k-1} j+k=\sum_{j=1}^{k} j=\frac{k(k+1)}{2}=m(k)$$

as desired.

□

We return to the event that v_i and v_{i+1}, arbitrary, become separated, which will involve removing enough edges to partition the segment of the cycle where v_i and v_{i+1} will be separated. We set up a regime under which we can analyze the potential cycle breaking events.

Picture a segment of the k-thickened cycle like this. For each integer $j \in \{0, \ldots k\}$, j is a representative of the equivalence class of the integers mod k. Depict the segment of the thickened cycle by placing the vertices whose indices are 0 (mod k) along a top row. In the second row, we arrange the vertices who are 1 (mod k), and in the third row the vertices in the 2 (mod k) class, etc., until the bottom row consists of the vertices in the $k-1$ (mod k) class. Then configure the edges. The resulting structure should resemble a regular lattice and is isomorphic to a segment of the thickened cycle.

$$\begin{array}{cccccccccccc}
v_0 & -- & -- & v_k & & & v_{2k} & & & & & \\
& v_1 & & & v_{k+1} & & & v_{2k+1} & & & & \\
& & \ddots & & & \ddots & & & & \ddots & & \\
& & & v_{k-1} & & & v_{2k-1} & -- & -- & v_{3k-1} & &
\end{array}$$

The edges should be configured according to the regular thickened k-cycle process.

We fix two anchor edges to remove, both belong to the span-k edge class. We fix the "top" span-k edge, between v_0 and v_k, and require that we remove that edge. In the critical cut, we would also remove the span-k edge on the bottom, between the vertices (v_{k-1}, v_{2k-1}), in the $k-1$ (mod k) modulus class. To investigate the other possible partitions other than the critical cut, let r a positive integer be the "displacement" value of the $k-1$ (mod k) edge, so that for each r, we partition the segment under the condition that we remove the edge between the vertices of index $(0, k)$ (the top span-k edge) and the edge between the vertices of index $((k-1+kr),(k-1+kr+k))$ (the displaced bottom edge in the figure), and we do not remove any other edges whose vertices are in the bottom row, the $k-1$ (mod k) equivalence class.

In order to partition v_i and v_{i+1}, our vertices of interest, while adhering to these conditions, we must remove *at least* kr additional edges, in order to "snag" the r-displaced edge. When selecting these extra kr edges to remove, notice that there are $k-2$ "rows" of vertices between the fixed edge and the r-displaced edge, and r columns of vertices. As we cut edges, for each of these $r(k-2)$ edges we can either traverse "over" or "under" the vertex. Letting

$\mathcal{A}_{v_0,r}$ be the event that the lattice is broken at v_1 and v_2 with our r-displaced edge, we have that

$$\mathbb{P}\left(\mathcal{A}_{v_0,r}\right) \leq 2^{(k-2)r} p^{kr},$$

since there are 2 states (over or under) for each vertex in the middle rows and each additional edge is removed with probability p. With the symmetry of the cycle, we consider all vertices v_0 through v_n over all displacements r as n grows. Then we have

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{v,r} \mathcal{A}_{v_i,r}\right) \leq np^m \sum_{r\geq 0} (2^{k-2}p^k)^r \to \frac{np^m}{(1-(2^{k-2}p^k)}, \tag{2}$$

since the sum is a convergent geometric series (with $p < 1/2$), and approaches 0 as long as $p = o(n^{1/m})$.

Hence, with the probability of the event that the outer cycle remains intact approaches 1 as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{v,r} \mathcal{A}_{v_i,r}\right)^C \to 1.$$

This proves Theorem 3.1. Next we prove and analogous result for removing a specified number of edges, as we do in the railroad model.

Lemma III.4. *For a graph $G_{k,p}$ drawn from the railroad model with n vertices and $k = 2$, as long as $p = o(n^{1/3})$, the outer lattice will remain intact with high probability.*

Note: Be aware that in the statement and proof of this lemma we consider the thickened cycle in the railroad model with n vertices, in contrast to the rest of the paper we treat the railroad model as having n^2.

Proof. Consider a vertex v_0. Recall from Lemma 3.2 that in order to separate v_0 and v_k we must separate two consecutive vertices, v_j and v_{j+1}, so WLOG let us look at separating v_1 and v_2.

Recall further that the critical cut, and thus the one that requires the least edges, to separate v_1 and v_2 consists of 3 edges; the set $\{e_{02}, e_{12}, e_{13}\}$. As in the proof of theorem 3.1, let r be the displacement value of the 1 mod 2 edge, where we consider additional separations that involve cutting both e_{02} and the edge between v_{1+2r} and v_{3+2r}. As before, in order to separate v_1 and v_2 we would need at least an additional $2r$ edges to be deleted in order to "snag" the displaced edge.

Let $\mathcal{A}_{v_0,r}$ be the event that a separation will occur of v_1 and v_2 under the condition that we must delete e_{02} and $e_{(1+2r)(3+2r)}$, where r is the displacement value. Then, over all vertices v_0 and all displacements r, and the symmetry of the construction if we were to fix the 1 mod 2 edges and displace the 0 mod 2 edges by r instead, we have

$$\mathbb{P}\left(\bigcup_{v,r}\mathcal{A}_{v,r}\right) \leq n\sum_{r\geq 0}\binom{2kn-(3+2r)}{\lfloor 2pn\rfloor-(3+2r)}\frac{1}{\binom{2kn}{\lfloor 2pn\rfloor}} \tag{3}$$

as n grows. The first binomial coefficient enumerates the number of ways to select the $\lfloor 2pn\rfloor$ edges to remove including the critical simple cut $\{e_{02}, e_{12}, e_{13}\}$ and the additional $2r$ intermediate edges that would need to be cut to satisfy the conditions, and the second binomial coefficient enumerates the total possible ways to select a subset of $\lfloor 2pn\rfloor$ edges. So each term of the sum encodes the probability of selecting one of the subsets of edges that will result in a separation according to the displacement criteria.

We see that

$$\frac{\binom{2kn-(3+2r)}{\lfloor 2pn\rfloor-(3+2r)}}{\binom{2kn}{\lfloor 2pn\rfloor}} = \frac{(2kn-(3+2r))!}{(\lfloor 2pn\rfloor-(3+2r))!(2kn-\lfloor 2pn\rfloor)!}\frac{\lfloor 2pn\rfloor!(2kn-\lfloor 2pn\rfloor)!}{(2kn)!} \tag{4}$$

which approaches

$$\frac{(2pn)^{3+2r}}{(2kn)^{3+2r}} = \frac{p^{3+2r}}{k^{3+2r}},$$

as $n\to\infty$. Thus we have

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcup_{v,r}\mathcal{A}_{v,r}\right) \leq n\sum_{r\geq 0}(p/k)^{3+2r} \sim \frac{np^3}{k^3}\sum_{r\geq 0}\left[(p/k)^2\right]^r = \frac{np^3}{k^3}\frac{1}{1-(p/k)^2}, \tag{5}$$

since the sum is a convergent geometric series. Hence, as long as $np^3\to 0$ as $n\to\infty$, we will have the probability of the lattice becoming disconnected as the result of some two vertices being separated bounded above as

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcup_{v,r}\mathcal{A}_{v,r}\right) \leq \frac{np^3}{k^3(1-(p/k)^2)} \to 0.$$

□

Applying this to a graph drawn from the railroad model with n^2 vertices, $k=2$, and $p=\lambda/2n$, we check that we meet the criteria. We have

$$p^3(n^2) = \frac{\lambda}{2n}^3 n^2 = \frac{\lambda^3}{8n} \to 0$$

as $n\to\infty$, so the probability that the pruned thickened cycle remains intact goes to 1.

B. Spacing the Train Stations Along the Outer Cycle

In this section we work toward the lead term of the bound in the main conjecture. We are interested in bounding the maximum gap M between train station vertices $v \in T$ along the thickened cycle, *where* $M = \max\{d(v_i, v_j) : v_i, v_j \in V(H), j = i+1\}$. We will prove that with high probability, the length of the maximum gap will be close to $\frac{1}{2}n \log n$. We conjecture but do not prove that this result extends to ensuring that the average maximum gap $\mathbb{E}M = \Theta(\frac{1}{2}n \log n)$, which is what we would like for our main result.

Recall that we select $2n$ vertices for the set of train stations T from the original n^2 vertices. Let $X_1, \ldots X_{2n}$ be random variables where each X_i is the length of the gap between the ith vertex and the $i-1$st vertex that is chosen in the set of train stations T. Let $M = \max\{X_1, \ldots X_{2n}\}$, the maximum gap. For large n, we treat the selection process as considering each vertex and independently adding it to T with probability $2/n$, so each $X_i \sim geom(2/n)$. Letting k be the length of a gap between consecutive vertices in T, we have $\mathbb{P}(X_i > k) = (1 - \frac{2}{n})^k$. We are interested in the maximum gap M so we look at

$$\mathbb{P}(M \leq k) = (P(X_1 \cap \ldots X_{2n}) \leq k) \tag{6}$$

$$= (P(X_1 \leq k))^{2n} = (1 - P(X_1 > k))^{2n} \tag{7}$$

$$= \left(1 - \left(1 - \frac{2}{n}\right)^k\right)^{2n}. \tag{8}$$

Lemma III.5. *Letting* $k = \frac{1}{2}n \log n + \frac{cn}{2}$, *then*

$$\mathbb{P}(M \leq k) = F(c),$$

as $n \to \infty$ *with*

$$F(c) = \exp(-2e^{-c})$$

so $F(c) \to 0$ *with* $c \to -\infty$ *and* $F(c) \to 1$ *as* $c \to \infty$.

Proof. As $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}(M \leq k) = \lim_{n\to\infty} \left(1 - \left(1 - \frac{2}{n}\right)^{\frac{1}{2}n \log n + \frac{cn}{2}}\right)^{2n} \tag{9}$$

$$= \lim_{n\to\infty} \left(1 - \left(1 - \frac{2}{n}\right)^{n\frac{1}{2}\log n} \left(1 - \frac{2}{n}\right)^{n\frac{c}{2}}\right)^{2n} \tag{10}$$

$$= \lim_{n\to\infty} \left(1 - \exp(-2)^{\frac{1}{2}\log n} \exp(-2)^{\frac{c}{2}}\right)^{2n} \tag{11}$$

$$= \lim_{n\to\infty} \left((1 - n^{-1}\exp(-c))^n\right)^2 = \exp(2e^{-c}). \tag{12}$$

□

We state the next lemma and our ideal application, proving the lemma and discussing the application, without proof.

Lemma III.6. *For any* $\epsilon > 0$,

$$\mathbb{P}(M \notin (\frac{1}{2} - \epsilon) n \log n, (\frac{1}{2} + \epsilon) n \log n \to 0.$$

Application We wish to show that, also for any $\epsilon > 0$, there exists an N for which, for $n \geq N$,

$$(\frac{1}{2} - \epsilon) n \log n < \mathbb{E}[M] < (\frac{1}{2} + \epsilon) n \log n.$$

Proof. For the first part of the lemma, we have $\mathbb{P}(M \notin (\frac{1}{2} - \epsilon) n \log n, (\frac{1}{2} + \epsilon) n \log n)$

$$= \mathbb{P}(M > (\frac{1}{2} + \epsilon) n \log n) + \mathbb{P}(M \leq \frac{1}{2} - \epsilon) n \log n) \tag{13}$$

$$= 1 - \mathbb{P}(M \leq (\frac{1}{2} + \epsilon) n \log n) + \mathbb{P}(M \leq \frac{1}{2} - \epsilon) n \log n) \tag{14}$$

$$= 1 - F(c_1) + F(c_2) \tag{15}$$

where $c_1 = 2\epsilon \log n$ and $c_2 = -2\epsilon \log n$. Then as n tends to ∞ we have $c_1 \to \infty$ and $c_2 \to -\infty$ as $n \to \infty$, so

$$\lim_{n \to \infty} 1 - F(c_1) + F(c_2) \tag{16}$$

$$= \lim_{n \to \infty} 1 - \exp(-2e^{-2\epsilon \log n}) + \exp(-2e^{2\epsilon \log n}) \tag{17}$$

$$= \lim_{n \to \infty} 1 - \exp(-2n^{-2\epsilon}) + \exp(-2n^{2\epsilon}) \to 1 - e^0 = 0. \tag{18}$$

□

So $\mathbb{P}(M \notin (\frac{1}{2} - \epsilon) n \log n, (\frac{1}{2} + \epsilon) n \log n) \to 0$.

To actually prove the application, we need a large deviation bound on the tail probabilities in Lemma 3.5.

C. Neighborhood Growth of the Train Stations

The $O(\log_\lambda 2n)$ component in the bound on the average distance between vertices in the railroad model is obtained from the expected distance between the train stations using the rail lines. A sharper result is given in [12], and we prove part of it here. We bound the probability that a particular pair of train stations will be more than distance $\log_\lambda 2n$ apart using the railline shortcuts.

Recall that with the probability of an edge given by $p = \lambda/2n$, $\lambda > 1$ a constant, the Erdős-Rényi random graph on T has a giant connected component with high probability [7],

[4]. We treat the growth of a train station's neighborhood in the Erdős-Rényi subgraph on T like a branching process in the early stages of growth, following [[?], [?]].

Let $v_0 \in T$ be an arbitrary train station. For each of the other $2n-1$ train stations, we introduce an edge with probability $\lambda/2n$. The expected number of neighboring train stations of v_0 is asymptotically λ. Simultaneously let $v_1 \in T$ be another train station, its neighborhood also growing the same way. Say that v_0 and v_1 are in the zeroeth generation and their neighboring stations are in the first generation. For the second generation, we introduce edges spawning from each train station in the first generation to any of the $2n$ train stations that have not been considered yet–since the edge probability is small and we are still in the early growth stages, asymptotically each train station in the first generation reaches λ others. In this way, for sufficiently small k, the size of the kth generation of v_0 (and that of v_1) is asymptotically λ^k.

Let S_k, S'_k be the set of vertices reachable in exactly k steps from v_0 (v_1). As stated above, $\mathbb{E}|S_k| \approx \lambda^k$.

Lemma III.7. *Let $v, v' \in T$ be fixed. Let S_k and S'_k be the sets of vertices reachable from v and v' respectively at distance exactly k. Let $A_{v,v'}$ be defined as the event that $S_k \cap S'_k \neq \emptyset$. Then after step $k = \frac{1}{2}\log 2n/\log\lambda$,*

$$\mathbb{P}\left(\overline{A_{v,v'}}\right) \leq e^{-1}$$

and

$$\mathbb{P}\left(A_{v,v'}\right) \geq 1 - e^{-1}.$$

Proof. With $k = \frac{1}{2}\log 2n/\log\lambda$, the expected sizes of S_k and S'_k will both be $\approx \lambda^k = \sqrt{2n}$. So, the event $\overline{A_{v,v'}}$ can be equivalently described as the event that S_k and S'_k are disjoint, having both of their $\sqrt{2n}$ elements being chosen from $2n$ vertices with no vertex being allowed to be selected for both S_k and S'_k. This can be expressed in terms of binomial coefficients, where, supposing we have already chosen the $\sqrt{2n}$ vertices in S_k, we have $\binom{2n-\sqrt{2n}}{\sqrt{2n}}$ ways we can choose S'_k so that $S_k \cap S'_k = \emptyset$. Hence, as $n \to \infty$,

$$\mathbb{P}\left(S_k \cap S'_k = \emptyset\right) = \frac{\binom{2n-\sqrt{2n}}{\sqrt{2n}}}{\binom{2n}{\sqrt{2n}}} \to e^{-1}, \tag{19}$$

By the binomial bound limit shown in Lemma .8 in the appendix.

Thus, we expect that after step $k = \frac{1}{2}\log 2n/\log\lambda$ that we have the neighborhoods of vertices reachable by v, v' begin to overlap with probability going to $1 - e^{-1}$, so the expected graph distance from v to v' is twice the number of steps we took from both train stations, $2\frac{1}{2}\log 2n/\log\lambda = \log_\lambda 2n$ with probability $1 - \exp(-1)$.

□

From here, we expect that the average length of a rail route along edges in the Erdős-Rényi graph on T is $O(\log_\lambda 2n)$. For a sharper result, see [12]. They advance prior results on the diameter in this range of $p(n)$ and show that for $p = \lambda/n$, $\lambda > 1$ fixed, the diameter of the largest connected component is

$$D(G_{n,\lambda/n}) = \frac{\log n}{\log \lambda} + 2\frac{\log n}{\log(1/\lambda^*)} + O_p(1)$$

where $\lambda^* < 1$ satisfies $\lambda e^{-\lambda} = \lambda^* e^{-\lambda^*}$ and $O_p(1)$ is an additive corrective term that is bounded in probability as $n \to \infty$.

CONCLUSION AND FURTHER WORK

Our approach to bounding the average shortest path length in the railroad model was based on bounding the quantity of interest separately in the two main geometric components of the graph: the pruned thickened cycle and the set of λn rail line shortcuts configured like a $G(2n, p(n))$ with $p(n) = \lambda/2n$. Bifurcating the approach in this way makes some simplifying assumptions about the structure of the graph. A more rigorous approach following methods from [12] could get us a sharper result on the diameter of the rail line shortcuts. Alternatively, we might approach the construction of the railroad model as a multi-type branching process as in [3], where we introduce edges of both types simultaneously from each vertex, the thick cycle edges in addition to the shortcuts, according to their respective probabilities.

With a sharper result on the average shortest path length in the railroad model, we are also interested in generalizing the model and its results to general $k > 2$. With a general k, we also become more interested in what the typical length of a traversal around the outer cycle is like once edges are removed. We believe the behavior of the outer cycle can be approached as a tiling problem, where for each path from one train station to another we have access to edges of length 1 up through k with the probability that those edges have not been removed. Ultimately, with a sharper result on the average path length, and a fuller understanding of the behavior of the outer cycle for general k, we are in a position to extend our results from the railroad model to a result on the Watts-Strogatz small-world characteristics.

APPENDIX

Lemma .8. *The binomial quotient bound limit.*

As $n \to \infty$,

$$\frac{\binom{n^2-an}{bn}}{\binom{n^2}{bn}} \to e^{-ab}.$$

Proof. Upper bound:

Expanding the binomials and dividing everything by n^2 after simplifying we have

$$\frac{(n^2-an)!(n^2-bn)!}{(n^2)!(n^2-an-bn)!} \tag{20}$$

$$= \frac{(n^2-bn)(n^2-bn-1)(n^2-bn-2)\dots(n^2-bn-(an-1))}{(n^2)(n^2-1)(n^2-2)\dots(n^2-(an-1))} \tag{21}$$

$$= \frac{(1-\frac{b}{n})(1-\frac{b}{n}-\frac{1}{n^2})(1-\frac{b}{n}-\frac{2}{n^2})\dots(1-\frac{b}{n}-\frac{(an-1)}{n^2})}{(1-\frac{1}{n^2})(1-\frac{2}{n^2})\dots(1-\frac{(an-1)}{n^2})}. \tag{22}$$

We apply the left and right inequalities $e^{-x(1+x)} \leq 1-x \leq e^{-x}$ for $x \leq \frac{1}{2}$ to equation (3), so that equation (3) is

$$\leq \frac{\exp(-\frac{b}{n})\exp(-\frac{b}{n}-\frac{1}{n^2})\exp(-\frac{b}{n}-\frac{2}{n^2})\dots\exp(-\frac{b}{n}-\frac{an-1}{n^2})}{\exp(-\frac{1}{n^2}(1+\frac{1}{n^2}))\exp(-\frac{2}{n^2}(1+\frac{2}{n^2}))\dots\exp(-\frac{an-1}{n^2}(1+\frac{an-1}{n^2}))}$$

$$= \frac{\exp(an(-\frac{b}{n}))\exp(-\frac{1}{n^2}\sum_{k=1}^{an-1}k)}{\exp(-\frac{1}{n^2}\sum_{k=1}^{an-1}k)\exp(-\frac{1}{n^4}\sum_{k=1}^{an-1}k^2)}$$

$$= \frac{\exp(-ab)}{\exp(-\frac{1}{n^4}\sum_{k=1}^{an-1}k^2)} \leq \exp(-ab),$$

since the sum of the first $an-1$ squares goes to $O(\frac{n^3}{6})$, forcing the exponennt in the denominator to 1 asymptotically.

Lower bound:

We bound the binomial quotient below by $\exp(-ab)$ by reversing the inequalities we use in the numerator and denominator:

$$\frac{\binom{n^2-an}{bn}}{\binom{n^2}{bn}} = \frac{(n^2-an)!(n^2-bn)!}{(n^2)!(n^2-an-bn)!} \tag{23}$$

$$= \frac{(n^2-bn)(n^2-bn-1)(n^2-bn-2)\dots(n^2-bn-(an-1))}{(n^2)(n^2-1)(n^2-2)\dots(n^2-(an-1))} \tag{24}$$

$$= \frac{(1-\frac{b}{n})(1-\frac{b}{n}-\frac{1}{n^2})(1-\frac{b}{n}-\frac{2}{n^2})\dots(1-\frac{b}{n}-\frac{(an-1)}{n^2})}{(1-\frac{1}{n^2})(1-\frac{2}{n^2})\dots(1-\frac{(an-1)}{n^2})}. \tag{25}$$

Reversing the inequalities we use we get the above bounded below by

$$\geq \frac{\exp(-ab)\exp(-ab/n)\exp(-\frac{2b}{n^3}\sum_{k=1}^{an-1}k)\exp(-\frac{1}{n^2}\sum_{k=1}^{an-1}k)\exp(-\frac{1}{n^4}\sum_{k=1}^{an-1}k^2)}{\exp(-\frac{1}{n^2}\sum_{k=1}^{an-1}k)} \tag{26}$$

$$= \exp(-ab)\exp(-\frac{ab}{n})\exp(-\frac{2b}{n^3}\sum_{k=1}^{an-1}k)\exp(-\frac{1}{n^4}\sum_{k=1}^{an-1}k^2) \tag{27}$$

$$= \exp(-ab)\exp(-\frac{ab}{n})\exp(-\frac{b(an)(an-1)}{n^3})\exp(-\frac{(an)(an-1)(2an-1)}{n^4}) \tag{28}$$

$$\geq \exp(-ab), \tag{29}$$

for all n, since each of the smaller order terms are smaller than 1. Hence, the binomial coefficient quotient goes to $\exp(-ab)$ as $n \to \infty$.

□

REFERENCES

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[1] Addario-Berry, L. and Lei, T., "The mixing time of the newman–watts small world," (2012), arXiv:1201.3795.

[2] Alon, N. and Spencer, J. H., *The probabilistic method*, 4th ed., Wiley Series in Discrete Mathematics and Optimization (John Wiley & Sons, Inc., Hoboken, NJ, 2016) pp. xiv+375.

[3] Barbour, A. D. and Reinert, G., Electron. J. Probab. **11**, no. 47, 1234 (2006).

[4] Bollobás, B., *Random graphs*, 2nd ed., Cambridge Studies in Advanced Mathematics, Vol. 73 (Cambridge University Press, Cambridge, 2001) pp. xviii+498.

[5] Bollobás, B. and Chung, F. R. K., SIAM J. Discrete Math. **1**, 328 (1988).

[6] Durrett, R., *Random graph dynamics*, Cambridge Series in Statistical and Probabilistic Mathematics, Vol. 20 (Cambridge University Press, Cambridge, 2010) pp. x+210.

[7] Erd″ os, P. and Rényi, A., Publ. Math. Debrecen **6**, 290 (1959).

[8] Newman, M. E. J., in *Handbook of graphs and networks* (Wiley-VCH, Weinheim, 2003) pp. 35–68.

[9] Newman, M. E. J. and Watts, D. J., Phys. Lett. A **263**, 341 (1999).

[10] Newman, M. E. J. and Watts, D. J., (1999), 10.1103/PhysRevE.60.7332, arXiv:cond-mat/9904419.

[11] Palmer, E. M., *Graphical evolution*, Wiley-Interscience Series in Discrete Mathematics (John Wiley & Sons, Ltd., Chichester, 1985) pp. xvii+177, an introduction to the theory of random graphs, A Wiley-Interscience Publication.

[12] Riordan, O. and Wormald, N., Combin. Probab. Comput. **19**, 835 (2010).

[13] Watts, D. J. and Strogatz, S. H., Nature **393**, 440 (1998).

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